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The time-fractional radiative transport equation—Continuous-time random walk, diffusion approximation, and Legendre-polynomial expansion

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We consider the radiative transport equation in which the time derivative is replaced by the Caputo derivative. Such fractional-order derivatives are related to anomalous transport and anomalous diffusion. In this paper we describe how the time-fractional radiative transport equation is obtained from continuous-time random walk and see how the equation is related to the time-fractional diffusion equation in the asymptotic limit. Then we solve the equation with Legendre-polynomial expansion. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4973441]

I. INTRODUCTION

Anomalous diffusion is often observed in nature.[31,33] For example, tracer particles flowing in an aquifer exhibits anomalous diffusion.[1] At the macroscopic scale after multiple scattering takes place, such anomalous diffusion is governed by fractional diffusion equations.[31,32,39] Considering the fact that the diffusion equation appears in the asymptotic limit of the radiative transport equation or the linear Boltzmann equation,[18] one can expect that at the mesoscopic scale there exist anomalous transport phenomena which are described by the fractional radiative transport equation. The use of the radiative transport equation was proposed for predicting the concentration of radionuclides in fractured rock underground.[40,41] If this happens, then its fractional version must appear just like the fractional diffusion equation shows up when the diffusion process takes place in a complex structure.

Let $\alpha \in (0,1)$ and $\sigma_t, \sigma_s \in (0,\infty)$ be constants determined by the medium under consideration. We suppose $\sigma_t > \sigma_s$. Let $v > 0$ be a constant speed. Let $u(x,\mu,t)$ $(x \in \mathbb{R}, \mu \in [-1,1], t \in [0,\infty))$ be the angular density. We consider the following initial-value problem for the time-fractional radiative transport equation:

$$\begin{cases} \partial_t^\alpha u(x,\mu,t) + v\mu\partial_x u(x,\mu,t) + \sigma_t u(x,\mu,t) = \sigma_s \int_{-1}^{1} p(\mu,\mu')u(x,\mu',t)\,d\mu', \\ \hfill u(x,\mu,0) = \delta(x)\delta(\mu-\mu_0), \end{cases} \tag{1}$$

where $\delta(\cdot)$ is the Dirac delta function and ∂_t^α is the Caputo fractional derivative,[3] which is defined by[35]

$$\partial_t^\alpha u(\cdot,\cdot,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_{t'} u(\cdot,\cdot,t')}{(t-t')^\alpha}\,dt', \qquad 0 < \alpha < 1,$$

with $\Gamma(\cdot)$ the Gamma function. Indeed, u in (1) is the fundamental solution of the time-fractional radiative transport equation. We note that recently ∂_t^α was redefined more generally using fractional Sobolev spaces.[10] Compared with the Riemann-Liouville derivative, the Caputo derivative is not singular at $t = 0$. Thus we can have the same initial condition in (1) and in the corresponding

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equation of the first derivative ∂_t. The phase function $p(\mu,\mu')$ satisfies

$$\int_{-1}^{1} p(\mu,\mu')\,d\mu' = 1, \qquad \forall\mu \in [-1,1].$$

Anomalous transport phenomena are in the transport regime when the distance of interest is not too large compared to the scattering mean free path v/σ_s, and as is shown below, the time-fractional diffusion equation is obtained from (1) in the asymptotic limit. The time-fractional diffusion equation has been intensively studied. In addition to several examples,[31,33] we point out that the behavior of water transport in granite was successfully reproduced by the random walk process with a power-law distribution.[11] It is proposed that if there are two porosities, the mass transport in fractured porous aquifer should be governed by the diffusion equation in which both ∂_t and ∂_t^α appear.[6] The Cauchy problem[4] and initial-boundary-value problem[24,26] were considered for the time-fractional diffusion equation. The maximum principle was established.[23] The technique of eigenfunction expansion was developed.[37] Numerical algorithms for the equation have been developed.[22] Moreover the standard time-fractional diffusion equation was generalized to equations with multiple Caputo derivatives[19,25] and distributed-order equations.[16,20] See the recent review by Jin and Rundell.[14]

The rest of the paper is organized as follows. In Section II, we obtain the time-fractional radiative transport equation from continuous-time random walk. In Section III, we see that the time-fractional diffusion equation emerges from the time-fractional radiative transport equation when absorption is small, propagation distance is large, and observation time is long. In Section IV, we express the solution to the time-fractional radiative transport equation in the form of Legendre polynomial expansion. In Section V, we numerically compute the solutions of the time-fractional radiative transport equation and of the time-fractional diffusion equation. Finally in Section VI, concluding remarks are made. The subtraction of the ballistic term is considered in the Appendix.

II. CONTINUOUS-TIME RANDOM WALK

We consider the continuous-time random walk whose jump probability density function $\varphi(x,t;\mu,\mu')$ ($x \in \mathbb{R}$, $t \in [0,\infty)$, $\mu,\mu' \in [-1,1]$) is given by

$$\varphi(x,t;\mu,\mu') = \left[\xi_s\delta(x)p(\mu,\mu') + (1-\xi_t)\delta(x-\mu r)\delta(\mu-\mu')\right] w(t), \tag{2}$$

where $\xi_t \in (0,1)$, $\xi_s \in (0,\xi_t)$, and $r > 0$ are some constants. The first term represents scattering and the second term in the square brackets of (2) is responsible for transport. The waiting time probability density function $w(t)$ is obtained as

$$(1-\xi_a)w(t) = \int_{-1}^{1}\int_{-\infty}^{\infty} \varphi(x,t;\mu,\mu')\,dx d\mu',$$

where $\xi_a = \xi_t - \xi_s > 0$ is the probability for absorption. The left-hand side of the above-mentioned equation shows the probability that the test particle is not absorbed in the medium and makes a jump after the time t.

Let $\eta(x,\mu,t)$ be the probability density function of just having arrived at position x at time t in direction μ. Let $P(x,\mu,t)$ be the probability density function of being at $(x,\mu,t) \in \mathbb{R}\times[-1,1]\times[0,\infty)$. We consider the following continuous-time random walk process:

$$\begin{cases} \eta(x,\mu,t) = \displaystyle\int_0^t\int_{-1}^{1}\int_{-\infty}^{\infty} \eta(x',\mu',t')\varphi(x-x',t-t';\mu,\mu')\,dx'd\mu'dt' + a(x,\mu)\delta(t), \\ \qquad\qquad\qquad P(x,\mu,t) = \displaystyle\int_0^t \eta(x,\mu,t')\Phi(t-t')\,dt', \end{cases}$$

where $a(x,\mu)$ is the initial value which is a function of x and μ, and $\Phi(t)$ is the cumulative probability of not having moved during t, which is given by

$$\Phi(t) = 1 - \int_0^t w(t')\,dt'.$$

By the Fourier-Laplace transform we have

$$(\mathcal{L}\mathcal{F}P)(k,\mu,s) = \int_0^\infty e^{-st}\int_{-\infty}^\infty e^{-ikx}P(x,\mu,t)\,dxdt$$
$$= (\mathcal{L}\mathcal{F}\eta)(k,\mu,s)(\mathcal{L}\Phi)(s),$$

where

$$(\mathcal{L}\Phi)(s) = \frac{1-(\mathcal{L}w)(s)}{s}.$$

Hence we obtain

$$(\mathcal{L}\mathcal{F}\eta)(k,\mu,s) = \left[\xi_s\int_{-1}^1 p(\mu,\mu')(\mathcal{L}\mathcal{F}\eta)(k,\mu',s)\,d\mu' + (1-\xi_t)(\mathcal{L}\mathcal{F}\eta)(k,\mu,s)e^{-i\mu rk}\right](\mathcal{L}w)(s) + (\mathcal{F}a)(k,\mu).$$

We consider small k and use

$$e^{-i\mu rk} \sim 1 - i\mu rk.$$

Thus we arrive at

$$\frac{1-(\mathcal{L}w)(s)}{(\mathcal{L}w)(s)}\left[(\mathcal{L}P)(x,\mu,s) - \frac{1}{s}P(x,\mu,0)\right]$$
$$= \xi_s\int_{-1}^1 p(\mu,\mu')(\mathcal{L}P)(x,\mu',s)\,d\mu' - \left[\xi_t + (1-\xi_t)r\mu\partial_x\right](\mathcal{L}P)(x,\mu,s).$$

Recalling $0<\alpha<1$, we have[35,38]

$$\left(\mathcal{L}\partial_t^\alpha f\right)(s) = s^\alpha\left(\mathcal{L}f\right)(s) - s^{\alpha-1}f(0).$$

Let us assume that the waiting time probability density function behaves as

$$(\mathcal{L}w)(s) \sim 1-(\tau s)^\alpha, \qquad 0 < s \ll \frac{1}{\tau}.$$

We introduce

$$\sigma_t = \frac{\xi_t}{\tau^\alpha}, \qquad \sigma_s = \frac{\xi_s}{\tau^\alpha}, \qquad v = \frac{(1-\xi_t)r}{\tau^\alpha}.$$

We asymptotically obtain

$$\partial_t^\alpha P(x,\mu,t) + v\mu\partial_x P(x,\mu,t) + \sigma_t P(x,\mu,t) = \sigma_s\int_{-1}^1 p(\mu,\mu')P(x,\mu',t)\,d\mu'.$$

This is (1).

Remark 2.1. In this section we implemented the effect of absorption in our random walk by introducing ξ_a. Such extension of the usual continuous-time random walk is done by Hornung, Berkowitz, and Barkai,[13] and by Henry, Langlands, and Wearne.[12] Indeed, we arrive at the same conclusion by instead writing (2) as

$$\varphi(x,t;\mu,\mu') = \left[\xi_s\delta(x)p(\mu,\mu') + (1-\xi_t)\delta(x-\mu r)\delta(\mu-\mu')\right]\frac{w(t)}{1-\xi_a},$$

with the waiting time probability density function $w(t)$ introduced as

$$w(t) = \int_{-1}^1\int_{-\infty}^\infty \varphi(x,t;\mu,\mu')\,dxd\mu'.$$

We can then give $\eta(x,\mu,t)$ and $P(x,\mu,t)$ as

$$\begin{cases} \eta(x,\mu,t) = (1-\xi_a)\displaystyle\int_0^t\int_{-1}^1\int_{-\infty}^\infty \eta(x',\mu',t')\varphi(x-x',t-t';\mu,\mu')\,dx'd\mu'dt' + a(x,\mu)\delta(t), \\ P(x,\mu,t) = (1-\xi_a)\displaystyle\int_0^t \eta(x,\mu,t')\Phi(t-t')\,dt'. \end{cases}$$

Note that $P(x,\mu,0) = (1-\xi_a)a(x,\mu)$. Thus the relation to the past work[12,13] becomes clearer.

III. DIFFUSION APPROXIMATION

Let us suppose that the ratio $\epsilon > 0$ of the mean free path to the propagation distance is small. We scale t, x as $t \to \epsilon^{2/\alpha} t$ and $x \to \epsilon x$. Furthermore we scale $\sigma_a \to \sigma_a/\epsilon^2$ assuming σ_a is small (recall $\sigma_a = \sigma_t - \sigma_s$). Although the radiative transport equation (1) has the Caputo derivative, we obtain the time-fractional diffusion equation by following the standard procedure.[2,18,36] In this section we assume that $p(\mu, \mu') = p(\mu', \mu)$. We can write the time-fractional radiative transport equation as

$$\epsilon^2 \partial_t^\alpha u(x,\mu,t) + \epsilon v \mu \partial_x u(x,\mu,t) + \left(\epsilon^2 \sigma_a + \sigma_s\right) u(x,\mu,t) = \sigma_s \int_{-1}^{1} p(\mu,\mu') u(x,\mu',t)\, d\mu'.$$

We write

$$u(x,\mu,t) = U_{\mathrm{DA}}(x,\mu,t) + \epsilon U_{\mathrm{DA}}^{(1)}(x,\mu,t) + \epsilon^2 U_{\mathrm{DA}}^{(2)}(x,\mu,t) + \cdots.$$

Let us collect terms of order ϵ^0. We obtain

$$\sigma_s U_{\mathrm{DA}}(x,\mu,t) = \sigma_s \int_{-1}^{1} p(\mu,\mu') U_{\mathrm{DA}}(x,\mu',t)\, d\mu'.$$

The above equation implies that U_{DA} is independent of μ; hereafter we write $U_{\mathrm{DA}}(x,\mu,t) = U_{\mathrm{DA}}(x,t)$. The terms of order ϵ^1 yield

$$v \mu \partial_x U_{\mathrm{DA}}(x,t) + \sigma_s U_{\mathrm{DA}}^{(1)}(x,\mu,t) = \sigma_s \int_{-1}^{1} p(\mu,\mu') U_{\mathrm{DA}}^{(1)}(x,\mu',t)\, d\mu'.$$

We obtain

$$U_{\mathrm{DA}}^{(1)}(x,\mu,t) = -\frac{v}{(1-\mathrm{g})\sigma_s} \mu \partial_x U_{\mathrm{DA}}(x,t),$$

where $\mathrm{g} \in (-1,1)$ satisfies

$$\mu \mathrm{g} = \int_{-1}^{1} \mu' p(\mu,\mu')\, d\mu'.$$

By collecting terms of order ϵ^2 we have

$$\begin{aligned} &\partial_t^\alpha U_{\mathrm{DA}}(x,t) + \mu \partial_x U_{\mathrm{DA}}^{(1)}(x,\mu,t) + \sigma_a U_{\mathrm{DA}}(x,t) + \sigma_s U_{\mathrm{DA}}^{(2)}(x,\mu,t) \\ &= \sigma_s \int_{-1}^{1} p(\mu,\mu') U_{\mathrm{DA}}^{(2)}(x,\mu',t)\, d\mu'. \end{aligned}$$

If we integrate the above equation over μ, we obtain

$$\partial_t^\alpha U_{\mathrm{DA}}(x,t) - D_0 \partial_x^2 U_{\mathrm{DA}}(x,t) + \sigma_a U_{\mathrm{DA}}(x,t) = 0, \tag{3}$$

where

$$D_0 = \frac{v}{3(1-\mathrm{g})\sigma_s}. \tag{4}$$

Thus the time-fractional diffusion equation is obtained in the asymptotic limit of (1).

One remark needs to be made. We have the second derivative for the spatial variable x in (3). In a similar setting, it is known that the space-fractional diffusion equation is obtained if the phase function decays with power-law as a function of the speed of propagating particles.[29,30]

IV. LEGENDRE-POLYNOMIAL EXPANSION

Let us suppose that $p(\mu, \mu')$ is given by

$$p(\mu,\mu') = \frac{1}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) P_l(\mu'),$$

where $L \geq 0$, and β_l $(l = 0, 1, \ldots, L)$ are positive constants such as $\beta_0 = 1$, $\beta_l < 2l + 1$ for $l \geq 1$. Here, $P_l(\mu)$ are the Legendre polynomials recursively given by

$$(l+1)P_{l+1}(\mu) = (2l+1)\mu P_l(\mu) - lP_{l-1}(\mu), \qquad P_1(\mu) = \mu, \qquad P_0(\mu) = 1, \qquad \mu \in [-1, 1].$$

In the time-independent case, an analytical solution of the space-fractional radiative transport equation was found.[15] In this section we solve (1). Let us expand u with Legendre polynomials.

$$(\mathcal{F}u)(k, \mu, t) = \sum_{l=0}^{\infty} \sqrt{2l+1}\, c_l(k, t; \mu_0) P_l(\mu). \tag{5}$$

We perform the Fourier transform in (1) and substitute (5). We have

$$\left(\partial_t^\alpha + ivk\mu + \sigma_t\right) \sum_{l=0}^{\infty} \sqrt{2l+1}\, c_l(k, t; \mu_0) P_l(\mu)$$
$$= \sigma_s \sum_{l=0}^{\infty} \sqrt{2l+1}\, c_l(k, t; \mu_0) \frac{\beta_l}{2l+1} P_l(\mu) \Theta(L - l).$$

Let us introduce

$$h_l = 2l + 1 - \frac{\sigma_s}{\sigma_t} \beta_l \Theta(L - l).$$

Let N $(\geq L)$ be an integer. We take projections with $P_l(\mu)$ $(l = 0, 1, \ldots, N)$ and obtain

$$\frac{ivkl}{\sqrt{4l^2 - 1}} c_{l-1} + \frac{ivk(l+1)}{\sqrt{4(l+1)^2 - 1}} c_{l+1} + \partial_t^\alpha c_l + \frac{\sigma_t h_l}{2l+1} c_l = 0,$$

where we used the recurrence relations and orthogonality relations of Legendre polynomials,

$$\mu P_l(\mu) = \frac{l+1}{2l+1} P_{l+1}(\mu) + \frac{l}{2l+1} P_{l-1}(\mu) \tag{6}$$

and

$$\int_{-1}^{1} P_l(\mu) P_{l'}(\mu)\, d\mu = \frac{2}{2l+1} \delta_{ll'}.$$

The above equation is expressed as

$$A(k)\mathbf{c}(k, t; \mu_0) + \partial_t^\alpha \mathbf{c}(k, t; \mu_0) = 0,$$

where $A(k)$ is an $(N+1) \times (N+1)$ matrix and $\mathbf{c}(k, t; \mu_0)$ is an $N + 1$ dimensional vector defined by

$$\{A(k)\}_{ll'} = \frac{ivkl}{\sqrt{4l^2 - 1}} \delta_{l-1, l'} + \frac{\sigma_t h_l}{2l+1} \delta_{l, l'} + \frac{ivk(l+1)}{\sqrt{4(l+1)^2 - 1}} \delta_{l+1, l'}, \tag{7}$$

$$\{\mathbf{c}(k, t; \mu_0)\}_l = c_l(k, t; \mu_0). \tag{8}$$

When the Legendre polynomial expansion is used, tridiagonal matrices such as $A(k)$ appear due to the three-term recurrence relation (6).[7,8,21,34] By taking the Laplace transform we have

$$(\mathcal{L}\mathbf{c})(k, s; \mu_0) = (A(k) + s^\alpha)^{-1} s^{\alpha - 1} \mathbf{c}(k, 0; \mu_0),$$

where we used

$$(\mathcal{L}\partial_t^\alpha \mathbf{c})(k, s; \mu_0) = s^\alpha (\mathcal{L}\mathbf{c})(k, s; \mu_0) - s^{\alpha-1} \mathbf{c}(k, 0; \mu_0), \qquad 0 < \alpha \leq 1.$$

Let us recall that the Mittag-Leffler function is given by[35]

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad z, \alpha \in \mathbb{C}, \quad \Re\alpha > 0,$$

and the Laplace transform is obtained as

$$\mathcal{L}\{E_\alpha(zt^\alpha);\, s\} = \frac{s^{\alpha-1}}{s^\alpha - z}, \qquad z, s, \alpha \in \mathbb{C}, \quad \Re s, \Re\alpha > 0, \quad \left|\frac{z}{s^\alpha}\right| < 1.$$

Thus we find

$$\mathbf{c}(k,t;\mu_0) = E_\alpha\left(-A(k)t^\alpha\right)\mathbf{c}(k,0;\mu_0).$$

Since $\delta(\mu-\mu_0) = \sum_{l=0}^{\infty}\frac{2l+1}{2}P_l(\mu)P_l(\mu_0)$, we obtain

$$\{\mathbf{c}(k,0;\mu_0)\}_l = \frac{\sqrt{2l+1}}{2}P_l(\mu_0).$$

Let $\lambda_n(k)$ and $\mathbf{v}_n(k)$ be the nth eigenvalue and eigenvector of the matrix $A(k)$. We can write $A(k)$ as

$$A(k) = Q(k)D(k)Q(k)^{-1},$$

where

$$Q(k) = \left(\mathbf{v}_0(k)\ \mathbf{v}_1(k)\ \cdots\ \mathbf{v}_N(k)\right), \qquad D(k) = \mathrm{diag}(\lambda_0(k),\lambda_1(k),\dots,\lambda_N(k)).$$

We have

$$\{A(k)\}_{ij} = \left\{Q(k)D(k)Q(k)^{-1}\right\}_{ij} = \sum_{n=0}^{N}\lambda_n(k)v_n^{(i)}(k)v_n^{(j)*}(k),$$

where $v_n^{(i)}(k)$ is the ith component of $\mathbf{v}_n(k)$. Therefore we can write

$$\{\mathbf{c}(k,t;\mu_0)\}_l = \sum_{j=0}^{N}\frac{\sqrt{2j+1}}{2}P_j(\mu_0)\sum_{n=0}^{N}v_n^{(l)}(k)v_n^{(j)*}(k)E_\alpha\left(-\lambda_n(k)t^\alpha\right).$$

Noting (8), Eq. (5) yields

$$\begin{aligned} u(x,\mu,t) &\approx u(x,\mu,t;N) \\ &:= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx}\sum_{l=0}^{N}\sqrt{2l+1}\,c_l(k,t;\mu_0)P_l(\mu)\,dk. \end{aligned} \tag{9}$$

Since k appears always as ik, we see

$$c_l(-k,t;\mu_0) = c_l(k,t;\mu_0)^*.$$

We obtain

$$\begin{aligned} u(x,\mu,t;N) = \sum_{l=0}^{N}\frac{\sqrt{2l+1}}{\pi}P_l(\mu) \\ \times\int_0^\infty\left[\cos(kx)\Re c_l(k,t;\mu_0) - \sin(kx)\Im c_l(k,t;\mu_0)\right]dk. \end{aligned} \tag{10}$$

Remark 4.1. Although in this section we directly calculated u in (10), indeed, it is possible to directly relate $u(x,\mu,t)$ to $u_1(x,\mu,t)$ which is the solution of (1) with $\alpha = 1$. Let $f_\alpha(t)$ be a function such that

$$(\mathcal{L}f_\alpha)(s) = e^{-s^\alpha}.$$

For example, we have

$$f_{1/2}(t) = \frac{t^{-3/2}}{2\sqrt{\pi}}e^{-1/(4t)}.$$

If we introduce

$$\varphi(\tau,t) = \frac{t}{\alpha\tau^{1+1/\alpha}}f_\alpha\left(\frac{t}{\tau^{1/\alpha}}\right),$$

we have

$$(\mathcal{L}\varphi)(\tau,s) = s^{\alpha-1}e^{-\tau s^\alpha}.$$

Let us consider the Laplace transform of u with respect to s and u_1 with respect to s^α. Assuming $u(x,\mu,0) = u_1(x,\mu,0)$, we obtain

$$\begin{cases} s^\alpha(\mathcal{L}u)(x,\mu,s) - s^{\alpha-1}u(x,\mu,0) + v\mu\partial_x(\mathcal{L}u)(x,\mu,s) + \sigma_t(\mathcal{L}u)(x,\mu,x) \\ \qquad = \sigma_s\int_{-1}^{1} p(\mu,\mu')(\mathcal{L}u)(x,\mu',s)\,d\mu', \\ s^\alpha(\mathcal{L}u_1)(x,\mu,s^\alpha) - u_1(x,\mu,0) + v\mu\partial_x(\mathcal{L}u_1)(x,\mu,s^\alpha) + \sigma_t(\mathcal{L}u_1)(x,\mu,s^\alpha) \\ \qquad = \sigma_s\int_{-1}^{1} p(\mu,\mu')(\mathcal{L}u_1)(x,\mu',s^\alpha)\,d\mu'. \end{cases}$$

The above equations imply

$$(\mathcal{L}u)(x,\mu,s) = s^{\alpha-1}(\mathcal{L}u_1)(x,\mu,s^\alpha) = \int_0^\infty u_1(x,\mu,t)s^{\alpha-1}e^{-\tau s^\alpha}\,d\tau.$$

Therefore u and u_1 are related as

$$u(x,\mu,t) = \int_0^\infty u_1(x,\mu,\tau)\varphi(\tau,t)\,d\tau.$$

This means that we can obtain u by integrating u_1, which is the solution of the first-order equation. The solution u is subordinated to the solution u_1.[17]

V. NUMERICAL CALCULATION

The energy density $U(x,t)$ is introduced as

$$U(x,t) = \int_{-1}^{1} u(x,\mu,t)\,d\mu.$$

Each N gives an approximated value of $U(x,t)$ as

$$U(x,t) \approx U(x,t;N),$$

where

$$U(x,t;N) = \int_{-1}^{1} u(x,\mu,t;N)\,d\mu.$$

We note that $U(x,t) = U(x,t;\infty)$. Let us calculate $U(x,t;N)$ for the initial condition

$$U(x,0;N) = \delta(x).$$

From (10) we obtain

$$\begin{aligned} U(x,t;N) &= \int_{-1}^{1}\int_{-1}^{1} u(x,\mu,t;N)\,d\mu d\mu_0 \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{ikx}\sum_{n=0}^{N}\left|v_n^{(0)}(k)\right|^2 E_\alpha\left(-\lambda_n(k)t^\alpha\right)\,dk \\ &= \frac{2}{\pi}\sum_{n=0}^{N}\int_0^\infty \left|v_n^{(0)}(k)\right|^2 \\ &\times\left(\cos(kx)\Re E_\alpha\left(-\lambda_n(k)t^\alpha\right) - \sin(kx)\Im E_\alpha\left(-\lambda_n(k)t^\alpha\right)\right)dk. \end{aligned}$$

In this section we set

$$v = 1, \qquad \sigma_a = 0, \qquad L = N = 1$$

and

$$\sigma_s = 10, \qquad \mathrm{g} = \frac{\beta_1}{3} = 0.9.$$

The matrix $A(k)$ in (7) is given by

$$A(k) = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & ik \\ ik & 2k_c \end{pmatrix},$$

where we introduced

$$k_c := \frac{\sqrt{3}}{2}\sigma_s(1-\mathrm{g}).$$

Its eigenvalues and eigenvectors are obtained as

$$\lambda(k) = \frac{k_c}{\sqrt{3}}\left(1 \pm \sqrt{1-\left(\frac{k}{k_c}\right)^2}\right)$$

and

$$\mathbf{v}(k) = \frac{1}{\sqrt{\mathcal{N}}}\begin{pmatrix} \frac{ik}{\sqrt{3}} \\ \lambda(k) \end{pmatrix}, \qquad \mathcal{N} = \begin{cases} \frac{2k_c^2}{3}\left(1 \pm \sqrt{1-\left(\frac{k}{k_c}\right)^2}\right), & |k| \le k_c, \\ \frac{2}{3}k^2, & |k| > k_c. \end{cases}$$

Thus we have

$$\left|v^{(0)}(k)\right|^2 = \begin{cases} \frac{1}{2}\left(1 \mp \sqrt{1-\left(\frac{k}{k_c}\right)^2}\right), & |k| \le k_c, \\ \frac{1}{2}, & |k| > k_c. \end{cases}$$

The energy density is written as

$$\begin{aligned} U(x,t;1) = \frac{1}{\pi}\int_0^{k_c} & \cos(kx) \\ & \times\left[\left(1-\sqrt{1-\left(\frac{k}{k_c}\right)^2}\right)E_\alpha\left(-\frac{k_c+\sqrt{k_c^2-k^2}}{\sqrt{3}}t^\alpha\right)\right. \\ & \left. + \left(1+\sqrt{1-\left(\frac{k}{k_c}\right)^2}\right)E_\alpha\left(-\frac{k_c-\sqrt{k_c^2-k^2}}{\sqrt{3}}t^\alpha\right)\right] dk \\ & + \frac{2}{\pi}\int_{k_c}^\infty \cos(kx)\Re E_\alpha\left(-\frac{k_c - i\sqrt{k^2-k_c^2}}{\sqrt{3}}t^\alpha\right) dk. \end{aligned} \tag{11}$$

In the diffusion approximation the energy density is given as follows. If the initial condition is given by

$$U_{\mathrm{DA}}(x,0) = \delta(x),$$

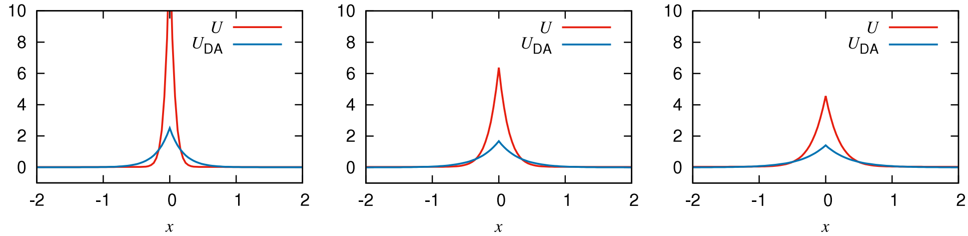

FIG. 1. Comparison of $U(x,t)$ and $U_{\mathrm{DA}}(x,t)$ as a function of x, from the left, for $t = 0.0001$, 0.0025, and 0.01, respectively, when $\alpha = 0.25$.

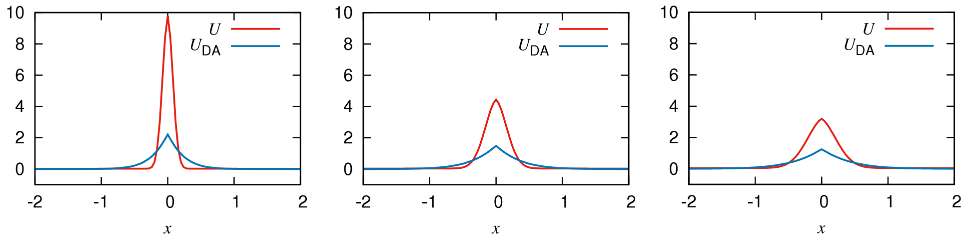

FIG. 2. Comparison of $U(x,t)$ and $U_{DA}(x,t)$ as a function of x, from the left, for $t = 0.01, 0.05$, and 0.1, respectively, when $\alpha = 0.5$.

we have[27,28]

$$U_{DA}(x,t) = \frac{1}{\pi}\int_0^{\infty} \cos(kx) E_\alpha(-D_0 k^2 t^\alpha)\, dk \tag{12}$$
$$= \frac{1}{\sqrt{D_0}} t^{-\frac{\alpha}{2}} M_{\alpha/2}\left(\frac{|x|}{\sqrt{D_0}\, t^{\alpha/2}}\right),$$

where $M_\alpha(z)$ is the M-Wright function defined by

$$M_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\Gamma(-\alpha(n+1)+1)}.$$

Equations (11) and (12) are implemented in Fortran. The numerical implementation of the Mittag-Lifter function relies on the algorithm by Gorenflo, Loutchko, and Luchko.[9] Although we saw in Section III that $U(x,t)$ asymptotically becomes $U_{DA}(x,t)$, they are different in general. In Figs. 1–3, we plot $U(x,t;1)$ and $U_{DA}(x,t)$ for $\alpha = 0.25, 0.5$, and 0.75, respectively. For all the cases, we see that $U(x,t;1)$ stays near the source at $x = 0$ for a relatively long time whereas $U_{DA}(x,t)$ broadens quickly. When $\alpha = 0.75$ we can see that $U(x,t;1)$ has two peaks. Such a double-peak structure shows up for $\alpha > 1$ in the case of the fractional diffusion equation.[28] This behavior can be understood from the relation[5]

$$E_\alpha(z) + E_\alpha(-z) = 2E_{2\alpha}(z^2), \qquad z \in \mathbb{C}.$$

For sufficiently large k, which corresponds to small x, we asymptotically have[35]

$$E_\alpha\left(-\frac{k_c - i\sqrt{k^2-k_c^2}}{\sqrt{3}} t^\alpha\right) \sim \frac{1}{\alpha}\exp\left[\left(-\frac{k_c - i\sqrt{k^2-k_c^2}}{\sqrt{3}} t^\alpha\right)^{1/\alpha}\right].$$

Hence in (11) we have

$$\Re E_\alpha\left(-\frac{k_c - i\sqrt{k^2-k_c^2}}{\sqrt{3}} t^\alpha\right) \sim \frac{1}{2\alpha}\exp\left[\left(i\frac{k}{\sqrt{3}}t^\alpha\right)^{1/\alpha}\right] + \frac{1}{2\alpha}\exp\left[\left(-i\frac{k}{\sqrt{3}}t^\alpha\right)^{1/\alpha}\right]$$
$$\sim \frac{1}{2}E_\alpha\left(i\frac{k}{\sqrt{3}}t^\alpha\right) + \frac{1}{2}E_\alpha\left(-i\frac{k}{\sqrt{3}}t^\alpha\right)$$

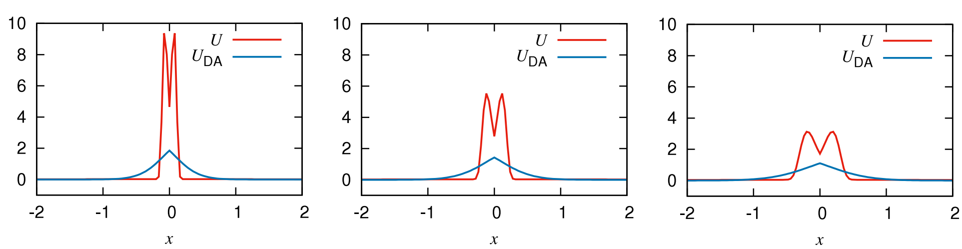

FIG. 3. Comparison of $U(x,t)$ and $U_{DA}(x,t)$ as a function of x, from the left, for $t = 0.05, 0.1$, and 0.2, respectively, when $\alpha = 0.75$.

$$= E_{2\alpha}\left(-\frac{1}{3}k^2 t^{2\alpha}\right).$$

The above calculation implies that the double-peak behavior for the fractional diffusion equation with $\alpha > 1$ can be seen for the fractional radiative transport equation with $\alpha > 1/2$.

VI. CONCLUDING REMARKS

One of the purposes of the present paper is to see the connection between the time-fractional radiative transport equation and the time-fractional diffusion equation. Roughly speaking, the time-fractional radiative transport equation of ∂_t^α behaves as the time-fractional diffusion equation of ∂_t^α for large x and behaves as the time-fractional diffusion equation of $\partial_t^{2\alpha}$ near $x = 0$ as is investigated in Sections III and V.

When $u(x,\mu,t)$ in (1) is expressed in the form of the collision expansion, the ballistic term is singular. If $u(x,\mu,t)$ itself is numerically computed, it is desirable to subtract the ballistic term. In a straightforward manner, we can extend the calculation in Section IV. This calculation is summarized in the Appendix.

APPENDIX: SUBTRACTION OF THE BALLISTIC TERM

Let us split $u(x,\mu,t)$ in (1) into the ballistic and scattered parts as

$$u(x,\mu,t) = u_b(x,\mu,t) + u_s(x,\mu,t),$$

where $u_b(x,\mu,t)$ and $u_s(x,\mu,t)$, respectively, satisfy

$$\begin{cases} \partial_t^\alpha u_b(x,\mu,t) + \mu\partial_x u_b(x,\mu,t) + \sigma_t u_b(x,\mu,t) = 0, \\ \hfill u_b(x,\mu,0) = \delta(x)\delta(\mu-\mu_0) \end{cases}$$

and

$$\begin{cases} \partial_t^\alpha u_s(x,\mu,t) + \mu\partial_x u_s(x,\mu,t) + \sigma_t u_s(x,\mu,t) = \sigma_s \int_{-1}^{1} p(\mu,\mu')u_s(x,\mu',t)\,d\mu' \\ \hfill + S(x,\mu,t), \\ \hfill u_s(x,\mu,0) = 0. \end{cases}$$

Here the source term for $u_s(x,\mu,t)$ is given by

$$S(x,\mu,t;\mu_0) = \sigma_s \int_{-1}^{1} p(\mu,\mu')u_b(x,\mu',t)\,d\mu'.$$

Noting that

$$(\mathcal{LF}u_b)(k,\mu,s) = \frac{s^{\alpha-1}}{s^\alpha + ik\mu + \sigma_t}\delta(\mu-\mu_0),$$

we obtain

$$u_b(x,\mu,t) = \frac{1}{2\pi}\delta(\mu-\mu_0)\int_{-\infty}^{\infty} e^{ikx}E_\alpha\left[-(ik\mu_0+\sigma_t)t^\alpha\right]\,dk$$

and

$$(\mathcal{LF}S)(k,\mu,s;\mu_0) = \sigma_s p(\mu,\mu_0)\frac{s^{\alpha-1}}{s^\alpha + ik\mu_0 + \sigma_t}.$$

Let us expand u_s with Legendre polynomials,

$$(\mathcal{F}u_s)(k,\mu,t) = \sum_{l=0}^{\infty}\sqrt{2l+1}\,c_l(k,t;\mu_0)P_l(\mu). \tag{A1}$$

For $0 \leq l \leq N$ we obtain

$$A(k)\mathbf{c}(k,t;\mu_0) + \partial_t^\alpha \mathbf{c}(k,t;\mu_0) = \mathbf{w}(k,t;\mu_0),$$

where $\mathbf{w}(k,t;\mu_0)$ is an $N+1$ dimensional vector defined by

$$\{\mathbf{w}(k,t;\mu_0)\}_l = \frac{\sqrt{2l+1}}{2}\int_{-1}^{1} P_l(\mu)(\mathcal{F}S)(k,\mu,t;\mu_0)\,d\mu.$$

By taking the Laplace transform we have

$$(\mathcal{L}\mathbf{c})(k,s;\mu_0) = (A(k)+s^\alpha)^{-1}\left[s^{\alpha-1}\mathbf{c}(k,0;\mu_0) + (\mathcal{L}\mathbf{w})(k,s;\mu_0)\right].$$

Let us express the Laplace transform of $\mathbf{w}(k,t;\mu_0)$ as

$$(\mathcal{L}\mathbf{w})(k,s;\mu_0) = \frac{s^{\alpha-1}}{s^\alpha + ik\mu_0 + \sigma_t}\mathbf{b}(\mu_0),$$

where

$$\{\mathbf{b}(\mu_0)\}_l = \frac{\sigma_s\beta_l}{2\sqrt{2l+1}}\Theta(L-l)P_l(\mu_0).$$

Using the relation

$$(A(k)-s^\alpha)^{-1}(\mathcal{L}\mathbf{w})(k,s;\mu_0)$$
$$= (A(k)+ik\mu_0+\sigma_t)^{-1}\left(\frac{s^{\alpha-1}}{s^\alpha+ik\mu_0+\sigma_t} - \frac{s^{\alpha-1}}{s^\alpha - A(k)}\right)\mathbf{b}(\mu_0),$$

we find

$$\mathbf{c}(k,t;\mu_0) = E_\alpha(-A(k)t^\alpha)\mathbf{c}(k,0;\mu_0)$$
$$+ (A(k)+ik\mu_0+\sigma_t)^{-1}\left[E_\alpha\left(-(ik\mu_0+\sigma_t)t^\alpha\right) - E_\alpha\left(A(k)t^\alpha\right)\right]\mathbf{b}(\mu_0).$$

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