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Global Lipschitz stability in determining coefficients of the radiative transport equation

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Abstract

In this paper, for the radiative transport equation, we study inverse problems of determining a time-independent scattering coefficient or total attenuation by boundary data on the complementary sub-boundary after making a one time input of a pair of a positive initial value and boundary data on a suitable sub-boundary. The main results are Lipschitz stability estimates. We can also prove the reverse inequalities, which means that our estimates for the inverse problems are the best possible. The proof is based on a Carleman estimate with a linear weight function.

Keywords: Lipschitz stability, transport equation, coefficient inverse problem, Carleman estimate

1. Radiative transport equation and main results

Let Ω be a bounded domain of \mathbb{R}^n , $n \geq 2$, with the C^1 -boundary $\partial\Omega$. We consider

$$P_0 u(x, v, t) + \sigma_t(x, v)u - \int_V k(x, v, v')u(x, v', t) dv' = 0, \quad (1.1)$$

$$x \in \Omega, v \in V, 0 < t < T,$$

$$u(x, v, 0) = a(x, v), \quad x \in \Omega, \quad v \in V, \quad (1.2)$$

$$u(x, v, t) = g(x, v, t), \quad 0 < t < T, \quad (x, v) \in \Gamma_-, \quad (1.3)$$

where

$$P_0 u(x, v, t) := \partial_t u(x, v, t) + v \cdot \nabla u(x, v, t).$$

Here and henceforth for $v, v' \in \mathbb{R}^n$, by $v \cdot v'$ we denote the scalar product in \mathbb{R}^n ; let $\nabla = \nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ and $V \subset \mathbb{R}^n$ be a bounded sub-domain or a measurable subset of $\{v \in \mathbb{R}^n; |v| = c\}$ with a constant $c > 0$. Here and henceforth \bar{V} denotes the closure of V .

Let $\nu(x)$ be the outward normal unit vector to $\partial\Omega$ at $x \in \partial\Omega$. We define Γ_+ and Γ_- by

$$\begin{aligned}\Gamma_+ &= \{(x, v) \in \partial\Omega \times V; (\nu(x) \cdot v) > 0\}, \\ \Gamma_- &= \{(x, v) \in \partial\Omega \times V; (\nu(x) \cdot v) \leq 0\}.\end{aligned}\tag{1.4}$$

The radiative transport equation or linear Boltzmann equation (1.1) governs non-interacting particles such as neutrons in a reactor [12, 18]. Equation (1.1) is also used for light propagating in random media such as biological tissue [1, 2], interstellar media [13] and atmospheres [38]. In (1.1), we let a real-valued function $u(x, v, t)$ denote the angular density of particles or the specific intensity of light at time $t \in (0, T)$ and position $x \in \Omega \subset \mathbb{R}^n$ with the velocity $v \in V$.

Let $\sigma_t(x, v)$ denote the total attenuation and satisfy

$$\sigma_t \in L^\infty(\Omega \times V)\tag{1.5}$$

and $k(x, v, v')$ be a scattering kernel which indicates the amount of particles scattering from a direction v' to a direction v at the position x . In this paper, we assume that k is independent of t and

$$k(x, v, v') \equiv \sigma_s(x, v)p(x, v, v'),\tag{1.6}$$

$$\begin{aligned}\sigma_s &\in L^\infty(\Omega \times V), \\ p &\in L^\infty(\Omega \times V \times V), > 0 \quad \text{in } \Omega \times V \times V.\end{aligned}\tag{1.7}$$

Throughout this paper, $p \in L^\infty(\Omega \times V \times V)$ is fixed, and σ_t or σ_s is unknown to be determined.

Remark 1.1. The parameters σ_t and k are called admissible [3, 15] if they satisfy $0 \leq \sigma_t \in L^\infty(\Omega \times V)$, $0 \leq k(x, \cdot, v') \in L^1(V)$ for almost all $x \in \Omega$ and $v' \in V$ and $\int_V k(\cdot, v, \cdot) dv \in L^\infty(\Omega \times V)$. Therefore, if we further assume that $\sigma_t, k \geq 0$ in $\Omega \times V$, then σ_t and k are admissible. For our arguments, we do not need the admissibility of σ_t and k .

Throughout this paper, we assume that there exist $\gamma \in \mathbb{R}^n, \neq 0$ and $\theta > 0$, such that

$$\bar{V} \subset \{v; (\gamma \cdot v) \geq \theta\}.\tag{1.8}$$

This means that we should restrict the distribution of v in a sector with the angle $< \pi$ with the vertex 0. We can consider the following experiment of optical tomography for example. That is, in (1.1), $u(x, v, t)$ is the specific intensity of light at time t and point $x \in \Omega$ with the velocity $v \in V \subset \{v \in \mathbb{R}^n; |v| = c\}$, where c is the speed of light. Let a slab-shaped box Ω be filled with a random medium such as a biological tissue. An array of sources on a face of Ω illuminates the medium, and the outgoing light is collected by an array of detectors on the other face. We suppose that the width of the box is thin in the sense that limited scattering takes place while the light travels from one side to the other. If the light is scattered in the forward direction when it collides with impurity, i.e., the direction does not change much by scattering, then the light stays within V when it reaches the other side. Thus, in this situation, we can assume that v is confined in V . That is, by taking γ in the direction perpendicular to the source and detector faces, we have $V = \{v \in \mathbb{R}^n; |v| = c, (\gamma \cdot v) \geq \theta\}$, where $\theta > 0$ is some constant. Thus, (1.8) is satisfied.

We rewrite (1.1) as

$$P_0 u(x, v, t) + \sigma_t(x, v)u - \sigma_s(x, v) \int_V p(x, v, v')u(x, v', t) dv' = 0.\tag{1.9}$$

In this paper, we consider inverse problems of determining total attenuation σ_t or a scattering coefficient σ_s in the radiative transport equation (1.9) by the boundary data

$$u(x, v, t), \quad (x, v) \in \Gamma_+, \quad 0 < t < T,$$

after setting up the initial value (1.2) and the boundary value (1.3) once. Our inverse problem is motivated for example by optical tomography, in which we recover σ_t and/or σ_s from boundary measurements (e.g., [1, 2]). An incident laser beam $g(x, v, t)$ enters the sample through the sub-boundary $\Gamma_- \times (0, T)$, and the outgoing light $u(x, v, t)$ is measured on the sub-boundary $\Gamma_+ \times (0, T)$. As is seen by theorems 1.1 and 1.2, we need the positivity for an initial value $a(x, v)$, and in remark 1.2 we describe such a setup.

We refer to works concerning inverse problems on the transport equation. Let us define the albedo operator as $\mathcal{A}[g] = u(x, v, t)$, $(x, v) \in \Gamma_+$, $0 < t < T$, where u is the solution to (1.1)–(1.3) with $a \equiv 0$. Choulli and Stefanov [15] proved the uniqueness of σ_t and $k = k(x, v, v')$. The corresponding stability is proved by the angularly averaged albedo operator [6] and by the full albedo operator [7]. For the inverse problems in [6, 7, 15], the input–output operation can be limited to the boundary, and the initial value can be zero, but one has to make infinite measurements.

Also, for the stationary transport equation, Choulli and Stefanov [16] and Tamasan [42] proved the uniqueness. The Hölder-type stability was obtained by Bal *et al* [4], Ball and Jollivet [5], Romanov [36, 37], Stefanov and Uhlmann [41] and Wang [45]. The non-uniqueness in the coefficient inverse problem with the albedo operator was characterized by gauge equivalent pairs in [40], and the Lipschitz stability for gauge equivalent classes was proved for the time-independent radiative transport equation in [34]. See also review articles [3, 39] for coefficient inverse problems for the radiative transport equation. Our paper discusses the determination of one coefficient σ_t or σ_s by a single measurement, while most of the above-mentioned papers are concerned with the simultaneous recovery of multiple parameters.

As for the simultaneous determination of both σ_t and σ_s by twice observations, see remark 1.3.

Klibanov and Pamyatnykh [30] proved the uniqueness of σ_t by the boundary values of u . The formulation in [30] is different from [6, 7, 15] and measures a single output on $\Gamma_+ \times (0, T)$ after choosing initial value and boundary data on $\Gamma_- \times (0, T)$.

In this paper, we adopt a similar formulation to [30] and consider the inverse problems of determining σ_s or σ_t by the boundary value on $\Gamma_+ \times (0, T)$ with a suitable single input of the initial value and boundary data on $\Gamma_- \times (0, T)$. Our main results are Lipschitz stability estimates in determining σ_s or σ_t . To the best knowledge of the authors, there are no publications on the Lipschitz stability with single measurement data related to the initial/boundary value problem (1.2), (1.3) and (1.9).

Our proof is based on the methodology by Bukhgeim and Klibanov [10] which uses a Carleman estimate, which is an L^2 -weighted estimate for solutions to the differential equation under consideration. Although the principle of our method is the same as [10] and [30], our key Carleman estimate (lemma 3.2) is of different character, and so we do not need any extension of the solution u to $(-T, T)$. On the other hand, [30] needs the extension of the solution u to $(-T, T)$, and so it further requires extra conditions for the unknown coefficients σ_t and the initial value a , such as $(a(x, v)\sigma_t(x, v))^2 = (a(x, -v)\sigma_t(x, -v))^2$ for $x \in \Omega$ and $v \in V$ in the case of $V = \{v; |v| = 1\}$. Klibanov and Pamyatnykh [29] proved the Lipschitz stability in determining σ_t when we consider the transport equation (1.9) for $-T < t < T$ with (1.2) and (1.2) is prescribed at an intermediate time $t = 0$ and is not an initial value. In [29], the extension argument is not necessary and the application of the relevant Carleman estimate is more direct.

The method by Bukhgeim and Klibanov [10] is useful for proving the uniqueness and the stability for coefficient inverse problems with a single measurement. The Carleman estimate dates back to [11] (see [20, 24, 33]). As for inverse problems by the Carleman estimate, we refer for example to [9, sections 1.10 and 1.11] and [21–23, 25–28, 31, 46].

On the other hand, Klibanov and Yamamoto [32] proved the exact controllability for the transport equation. Prilepko and Ivankov [35] discussed an inverse problem of determining a t -function in the case where σ_t depends on x, v, t . In the case of $k \equiv 0$, Gaitan and Ouzzane [19] proved the Lipschitz stability for σ_t using the method in [29, 30]. However, the argument requires more care in our case $k \neq 0$.

Our arguments in the case $V \subset \{v \in \mathbb{R}^n; |v| = c\}$ is the same as in the case of the sub-domain $V \subset \mathbb{R}^n$. Therefore, henceforth we assume that V is a sub-domain in \mathbb{R}^n . Throughout this paper, $H^m(\Omega)$ denotes usual Sobolev spaces. We set

$$X = H^1(0, T; L^\infty(\Omega \times V)) \cap H^2(0, T; L^2(\Omega \times V)).$$

For an arbitrarily fixed constant $M > 0$, we set

$$\mathcal{U} = \{u \in X; \|u\|_X + \|\nabla u\|_{H^1(0, T; L^2(\Omega \times V))} \leq M\}. \quad (1.10)$$

By the geometric configuration assumption (1.8) of V , we can choose $\gamma \in \mathbb{R}^n$ such that

$$\min_{v \in \bar{V}} (\gamma \cdot v) > 0.$$

We recall that $p \in L^\infty(\Omega \times V \times (0, T))$ is given.

Now we are ready to state our main results.

Theorem 1.1 (Determination of σ_t). *Let $u^i = u(\sigma_t^i)(x, v, t)$, $i = 1, 2$ be the solutions to the transport equation:*

$$\begin{aligned} \partial_t u(x, v, t) + v \cdot \nabla u + \sigma_t^i(x, v)u \\ - \sigma_s(x, v) \int_V p(x, v, v') u(x, v', t) dv' &= 0, \quad x \in \Omega, v \in V, 0 < t < T, \\ u(x, v, 0) &= a_i(x, v), \quad x \in \Omega, v \in V, \\ u &= g \quad \text{on } \Gamma_- \times (0, T). \end{aligned}$$

Let $u^i \in \mathcal{U}$ and $\|\sigma_t^i\|_{L^\infty(\Omega \times V)}, \|\sigma_s\|_{L^\infty(\Omega \times V)} \leq M$. Then we assume that

$$T > \frac{\max_{x \in \bar{\Omega}} (\gamma \cdot x) - \min_{x \in \bar{\Omega}} (\gamma \cdot x)}{\min_{v \in \bar{V}} (\gamma \cdot v)} \quad (1.11)$$

and there exists a constant $a_0 > 0$, such that

$$a_1(x, v) \geq a_0 \quad \text{or} \quad a_2(x, v) \geq a_0, \quad \text{almost all } (x, v) \in \Omega \times V. \quad (1.12)$$

Then there exists a constant $C = C(M, a_0) > 0$, such that

$$\left\{ \begin{aligned} \|\sigma_t^1 - \sigma_t^2\|_{L^2(\Omega \times V)} &\leq C \left(\int_0^T \int_{\Gamma_+} (v(x) \cdot v) |\partial_t (u^1 - u^2)(x, v, t)|^2 dS dv dt \right)^{1/2} \\ &\quad + C (\|a_1 - a_2\|_{L^2(\Omega \times V)} + \|\nabla a_1 - \nabla a_2\|_{L^2(\Omega \times V)}), \\ \left(\int_0^T \int_{\Gamma_+} (v(x) \cdot v) |\partial_t (u^1 - u^2)(x, v, t)|^2 dS dv dt \right)^{1/2} \\ &\leq C (\|\sigma_t^1 - \sigma_t^2\|_{L^2(\Omega \times V)} + \|a_1 - a_2\|_{L^2(\Omega \times V)} + \|\nabla a_1 - \nabla a_2\|_{L^2(\Omega \times V)}). \end{aligned} \right. \quad (1.13)$$

Here, $C(M, a_0) \rightarrow \infty$ as $M \rightarrow \infty$ or $a_0 \rightarrow 0$.

In particular, if we assume $a_1 = a_2$ in $\Omega \times V$, then we have a two-sided estimate:

$$C^{-1} \left(\int_0^T \int_{\Gamma_+} (v(x) \cdot v) |\partial_t(u^1 - u^2)(x, v, t)|^2 dS dv dt \right)^{1/2} \leq \| \sigma_t^1 - \sigma_t^2 \|_{L^2(\Omega \times V)}$$

$$\leq C \left(\int_0^T \int_{\Gamma_+} (v(x) \cdot v) |\partial_t(u^1 - u^2)(x, v, t)|^2 dS dv dt \right)^{1/2}.$$

This means that the choice of the norm of the boundary data on $\Gamma_+ \times (0, T)$ is the best possible for our inverse problem.

In this theorem, we assume the regularity $u^i \in \mathcal{U}, i = 1, 2$, for the solutions to the forward problem (1.9) and (1.2)–(1.3). We can prove it in terms of the regularity and compatibility conditions of the initial value a and the boundary value g and omit the details. In the following theorems, we assume the same regularity assumption on u^i . As for the forward problem of the transport equation, we refer for example to [8, 17, 35, 43, 44]. The dependence of the constant $C(M, a_0)$ in (1.13) on M and a_0 can be given explicitly, but we need lengthy arguments and omit details. In theorem 1.1, as is seen from the proof, we need not assume (1.6).

Theorem 1.2 (Determination of σ_s). *Let $u^i = u(\sigma_s^i)(x, v, t), i = 1, 2$, be the solution to the transport equation:*

$$\begin{aligned} & \partial_t u(x, v, t) + v \cdot \nabla u + \sigma_t(x, v)u \\ & - \sigma_s^i(x, v) \int_V p(x, v, v') u(x, v', t) dv' = 0, \quad x \in \Omega, v \in V, 0 < t < T, \\ & u(x, v, 0) = a_i(x, v), \quad x \in \Omega, v \in V, \\ & u = g \quad \text{on } \Gamma_- \times (0, T). \end{aligned}$$

Let $u^i \in \mathcal{U}$ and $\| \sigma_t \|_{L^\infty(\Omega \times V)}, \| \sigma_s^i \|_{L^\infty(\Omega \times V)} \leq M, i = 1, 2$, and we assume (1.11) and (1.12). Then there exists a constant $C = C(M, a_0) > 0$, such that

$$\left\{ \begin{aligned} & \| \sigma_s^1 - \sigma_s^2 \|_{L^2(\Omega \times V)} \leq C \left(\int_0^T \int_{\Gamma_+} (v(x) \cdot v) |\partial_t(u^1 - u^2)(x, v, t)|^2 dS dv dt \right)^{1/2} \\ & + C(\| a_1 - a_2 \|_{L^2(\Omega \times V)} + \| \nabla a_1 - \nabla a_2 \|_{L^2(\Omega \times V)}), \\ & \left(\int_0^T \int_{\Gamma_+} (v(x) \cdot v) |\partial_t(u^1 - u^2)(x, v, t)|^2 dS dv dt \right)^{1/2} \\ & \leq C(\| \sigma_s^1 - \sigma_s^2 \|_{L^2(\Omega \times V)} + \| a_1 - a_2 \|_{L^2(\Omega \times V)} + \| \nabla a_1 - \nabla a_2 \|_{L^2(\Omega \times V)}). \end{aligned} \right. \tag{1.14}$$

Here, $C(M, a_0) \rightarrow \infty$ as $M \rightarrow \infty$ or $a_0 \rightarrow 0$.

In (1.13) and (1.14), the second inequalities show the Lipschitz stability for the inverse problems, while the first inequalities assert the stability for the forward problems. Thus, we obtain both-sided estimates, and so the estimates for the inverse problems are the best possible.

Remark 1.2. Our inverse problem is for the well-posed initial boundary value problem, and we have to observe the initial values a_1, a_2 . Moreover, for the best possible Lipschitz stability for the inverse problems by a single measurement, we need not observe both initial values but need to set up one of the initial values in order to satisfy positivity (1.12). The positivity condition is restricting but can be achieved for example as follows for theorem 1.1. We assume that Ω is strictly convex and σ_t^1 is known, while σ_t^2 is unknown and we would like to estimate the deviation $\sigma_t^2 - \sigma_t^1$ around σ_t^1 . We consider the transport equations for $\sigma_t^i, i = 1, 2$, over the time interval $(-T_0, T)$ with some $T_0 > 0$. We arbitrarily choose $\tilde{a}(x, v)$ as the initial value of u^1 at $t = -T_0: u^1(x, v, -T_0) = \tilde{a}(x, v), x \in \Omega, v \in V$. We arbitrarily fix a constant $a_0 > 0$ and $a_1 \in L^2(\Omega \times V)$ satisfying $a_1 > a_0$ almost everywhere in $\Omega \times V$. By the exact controllability

result [32], if $T_0 > 0$ is sufficiently large, then we can find a function g_0 belonging to some weighted L^2 -space in $\Gamma_- \times (-T_0, 0)$, such that $u^1(x, v, 0) = a_1(x, v)$, $x \in \Omega$, $v \in V$. Here,

$$\partial_t u^1 + v \cdot \nabla u^1 + \sigma_t^1 u^1 - \sigma_s \int_V p(x, v, v') u^1(x, v', t) dv' = 0 \quad \text{in } \Omega \times V \times (-T_0, 0),$$

$$u^1(x, v, -T_0) = \tilde{a}(x, v), \quad x \in \Omega, \quad v \in V$$

and

$$u^1(x, v, t) = g_0(x, v, t), \quad (x, v, t) \in \Gamma_- \times (-T_0, 0).$$

Extending g_0 to $(-T_0, T)$ and denoting it by g , we consider the transport equations in $(0, T)$ for σ_t^i , $i = 1, 2$ with the same boundary value g on $\Gamma_- \times (0, T)$ and initial values a_i at $t = 0$. Then $a_1 > a_0$ almost everywhere in $\Omega \times V$ is satisfied. In other words, for u^2 satisfying the transport equation with σ_t^2 in $\Omega \times V \times (-T_0, T)$ with the boundary value g on $\Gamma_- \times (-T_0, T)$, we are requested to observe $a_2 := u^2(\cdot, \cdot, 0)$ in $\Omega \times V$ and u^2 on $\Gamma_+ \times (0, T)$. We note that we need not know the original initial value $u^2(\cdot, \cdot, -T_0)$, but the non-stationary transport equation with σ_t^2 should start at $t = -T_0$. In [32], the case of $V = \{v \in \mathbb{R}^n; |v| = 1\}$ is considered, and the modification to our case (1.8) is possible, but we omit the details. For realizing the positivity of solutions at some moment, see also [14, 47]. For theorem 1.2, we can argue similarly.

Moreover, we have to assume (1.11), that is, the observation time T should be sufficiently large. The principal part of the transport equation (1.1) is a hyperbolic operator of first order, which means that the transport equation has a finite propagation speed. Thus, we cannot avoid a condition like (1.11). Otherwise, we cannot detect the sufficient information of the coefficients in $\Omega \times V$ only by data on the boundary.

Remark 1.3. In order to determine σ_t and σ_s simultaneously, we choose two initial values a , b which satisfy for example

$$\det \begin{pmatrix} a & \int_V p(x, v, v') a(x, v') dv' \\ b & \int_V p(x, v, v') b(x, v') dv' \end{pmatrix} \neq 0.$$

The proof is similar to theorems 1.1 and 1.2 but we omit. The setup for such a, b is more complicated than that in theorem 1.1. For realizing such a, b , we can apply the exact controllability result by Klibanov and Yamamoto [32], but we omit the details.

In order to prove theorems 1.1 and 1.2, it is sufficient to prove the linearized inverse problem below.

Theorem 1.3. *We consider*

$$\begin{aligned} \partial_t u + v \cdot \nabla u + \sigma_t u - \int_V k(x, v, v') u(x, v', t) dv' \\ = f(x, v) R(x, v, t), \quad x \in \Omega, \quad v \in V, \quad 0 < t < T, \end{aligned}$$

$$u(x, v, 0) = a(x, v), \quad x \in \Omega, \quad v \in V.$$

We assume

$$k \in L^\infty(\Omega \times V \times V), \quad R, \partial_t R \in L^2(0, T; L^\infty(\Omega \times V)), \quad \sigma_t, \sigma_s \in L^\infty(\Omega \times V), \quad (1.15)$$

and

$$u \in \mathcal{U}. \quad (1.16)$$

For an arbitrarily fixed constant $a_0 > 0$, we further assume

$$R(x, v, 0) > a_0, \quad \text{almost all } (x, v) \in \Omega \times V \tag{1.17}$$

and

$$T > \frac{\max_{x \in \overline{\Omega}}(\gamma \cdot x) - \min_{x \in \overline{\Omega}}(\gamma \cdot x)}{\min_{v \in \overline{V}}(\gamma \cdot v)}. \tag{1.18}$$

There exists a constant $C > 0$, which depends on $\|\sigma_t\|_{L^\infty(\Omega \times V)}$, $\|k\|_{L^\infty(\Omega \times V \times V)}$ and $\|R\|_{H^1(0, T; L^\infty(\Omega \times V))}$, such that

$$\|f\|_{L^2(\Omega \times V)} \leq C \left(\int_0^T \int_{\partial\Omega} \int_V |(v \cdot v)| |\partial_t u|^2 \, dv \, dS \, dt \right)^{1/2} + C(\|a\|_{L^2(\Omega \times V)} + \|\nabla a\|_{L^2(\Omega \times V)}) \tag{1.19}$$

for all $f \in L^2(\Omega \times V)$.

Theorem 1.4. We assume (1.15)–(1.18). If $u = 0$ on $\Gamma_- \times (0, T)$ in theorem 1.3, then there exists a constant $C > 0$, which depends on $\|\sigma_t\|_{L^\infty(\Omega \times V)}$, $\|k\|_{L^\infty(\Omega \times V \times V)}$ and $\|R\|_{H^1(0, T; L^\infty(\Omega \times V))}$, such that

$$\|f\|_{L^2(\Omega \times V)} \leq C \left(\int_0^T \int_{\Gamma_+} (v \cdot v) |\partial_t u|^2 \, dS \, dv \, dt \right)^{1/2} + C(\|a\|_{L^2(\Omega \times V)} + \|\nabla a\|_{L^2(\Omega \times V)})$$

and

$$\left(\int_0^T \int_{\Gamma_+} (v \cdot v) |\partial_t u|^2 \, dS \, dv \, dt \right)^{1/2} \leq C(\|f\|_{L^2(\Omega \times V)} + \|a\|_{L^2(\Omega \times V)} + \|\nabla a\|_{L^2(\Omega \times V)})$$

for any $f \in L^2(\Omega \times V)$.

In fact, for the proof of theorem 1.1, assuming that $a_1 > a_0$ almost everywhere in $\Omega \times V$ and setting $u = u^2 - u^1$, $f = \sigma_t^2 - \sigma_t^1$, $a = a_2 - a_1$ and $R = -u^1$, we have the above-linearized inverse problem. By the regularity assumption of u^1 , u^2 and $\|\sigma_t^i\|_{L^\infty(\Omega \times V)}$, $\|\sigma_s^i\|_{L^\infty(\Omega \times V)} \leq M$, we can apply theorem 1.4 to obtain the conclusion (1.13). We can similarly derive theorem 1.2 from theorem 1.4.

Without the assumption $u^1 = u^2 = g$ on $\Gamma_- \times (0, T)$, by (1.19) we can obtain the Lipschitz stability in determining σ_t or σ_s :

$$\begin{aligned} \|\sigma_t^1 - \sigma_t^2\|_{L^2(\Omega \times V)} &\leq C \left(\int_0^T \int_{\partial\Omega} \int_V |(v \cdot v)| |\partial_t (u^1 - u^2)|^2 \, dS \, dv \, dt \right)^{1/2} \\ &\quad + C(\|a\|_{L^2(\Omega \times V)} + \|\nabla a\|_{L^2(\Omega \times V)}) \end{aligned}$$

or

$$\begin{aligned} \|\sigma_s^1 - \sigma_s^2\|_{L^2(\Omega \times V)} &\leq C \left(\int_0^T \int_{\partial\Omega} \int_V |(v \cdot v)| |\partial_t (u^1 - u^2)|^2 \, dS \, dv \, dt \right)^{1/2} \\ &\quad + C(\|a\|_{L^2(\Omega \times V)} + \|\nabla a\|_{L^2(\Omega \times V)}). \end{aligned}$$

The paper is composed of five sections. In section 2, we prove theorem 1.4 assuming that theorem 1.3 is proved. In section 3, we prove a key Carleman estimate and in section 4, we complete the proof of theorem 1.3. Section 5 gives concluding remarks.

2. Proof of theorem 1.4

Henceforth, in this section, $C > 0$ denotes generic constants which are independent of f .

Lemma 2.1. *Under the assumptions in theorem 1.3, for u satisfying the radiative transport equation with the initial value a and the right-hand side $f(x, v)R(x, v, t)$, there exists a constant $C > 0$, which depends on $\|\sigma_t\|_{L^\infty(\Omega \times V)}$ and $\|k\|_{L^\infty(\Omega \times V \times V)}$, such that*

$$\begin{aligned} \int_{\Omega} \int_V |\partial_t u(x, v, t)|^2 dv dx &\leq C(\|f\|_{L^2(\Omega \times V)}^2 + \|a\|_{L^2(\Omega \times V)}^2 + \|\nabla a\|_{L^2(\Omega \times V)}^2) \\ &\quad + C \int_0^T \int_{\Gamma_-} |(v \cdot \nu)| |\partial_t u|^2 dS dv dt \end{aligned} \quad (2.1)$$

for $0 \leq t \leq T$ and

$$\begin{aligned} \int_0^T \int_{\Gamma_+} |(v \cdot \nu)| |\partial_t u|^2 dS dv dt &\leq C(\|f\|_{L^2(\Omega \times V)}^2 + \|a\|_{L^2(\Omega \times V)}^2 + \|\nabla a\|_{L^2(\Omega \times V)}^2) \\ &\quad + C \int_0^T \int_{\Gamma_-} |(v \cdot \nu)| |\partial_t u|^2 dS dv dt. \end{aligned} \quad (2.2)$$

Theorem 1.4 is obtained from lemma 2.1 and theorem 1.3. In fact, the first inequality in theorem 1.4 follows from theorem 1.3 with $u = 0$ on Γ_- , while the second inequality is derived from (2.2). Thus, the rest part of this paper is devoted to the proofs of lemma 2.1 and theorem 1.3.

Proof of lemma 2.1. Taking the t -derivative of the transport equation, we have $\partial_t(\partial_t u) + v \cdot \nabla(\partial_t u) + \sigma_t(\partial_t u) - \int_V k(\partial_t u) dv' = f(x, v)\partial_t R$ with $(\partial_t u)(x, v, 0) = f(x, v)R(x, v, 0)$. Fixing $t \in (0, T)$ arbitrarily, multiplying this equation by $2\partial_t u$ and integrating over $\Omega \times V$, we have

$$\begin{aligned} \partial_t \int_{\Omega} \int_V |\partial_t u(x, v, t)|^2 dv dx &+ \int_{\Omega} \int_V v \cdot \nabla(|\partial_t u|^2) dv dx + 2 \int_{\Omega} \int_V \sigma_t |\partial_t u|^2 dv dx \\ &- 2 \int_{\Omega} \int_V \left(\int_V k(x, v, v') \partial_t u(x, v', t) dv' \right) \partial_t u(x, v, t) dv dx \\ &= 2 \int_{\Omega} \int_V f(\partial_t R) \partial_t u dv dx. \end{aligned}$$

Setting $E(t) = \int_{\Omega} \int_V |\partial_t u(x, v, t)|^2 dv dx$ and integrating the second term on the left-hand side, we obtain

$$\begin{aligned} E'(t) &= - \int_{\partial\Omega} \int_V (v \cdot \nu) |\partial_t u|^2 dv dS - 2 \int_{\Omega} \int_V \sigma_t |\partial_t u|^2 dv dx \\ &\quad + 2 \int_{\Omega} \int_V \left(\int_V k(x, v, v') \partial_t u(x, v', t) dv' \right) \partial_t u(x, v, t) dv dx \\ &\quad + 2 \int_{\Omega} \int_V f(\partial_t R) \partial_t u dv dx. \end{aligned}$$

Therefore, noting that $2 \int_{\Omega} \int_V |f(\partial_t R) \partial_t u| dv dx \leq \int_{\Omega} \int_V |f|^2 |\partial_t R|^2 dv dx + \int_{\Omega} \int_V |\partial_t u|^2 dv dx$, integrating over $(0, t)$ and using $k, \sigma_t \in L^\infty$, we have

$$\begin{aligned} E(t) - E(0) &= - \int_0^t \left(\int_{\Gamma_+} + \int_{\Gamma_-} \right) (v \cdot \nu) |\partial_t u|^2 dS dv dt - 2 \int_0^t \int_{\Omega} \int_V \sigma_t |\partial_t u|^2 dv dx dt \\ &\quad + 2 \int_0^t \int_{\Omega} \int_V \left(\int_V k(x, v, v') \partial_t u(x, v', t) dv' \right) \partial_t u(x, v, t) dv dx dt \end{aligned}$$

$$\begin{aligned}
& + 2 \int_0^t \int_{\Omega} \int_V f(\partial_t R) \partial_t u \, dv \, dx \, dt \\
\leq & - \int_0^t \int_{\Gamma_+} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt - \int_0^t \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt \\
& + C \int_0^t E(\eta) \, d\eta + C \|f\|_{L^2(\Omega \times V)}^2
\end{aligned} \tag{2.3}$$

for $0 \leq t \leq T$. Here by the Cauchy–Schwarz inequality and $k \in L^\infty(\Omega \times V \times V)$, we also used

$$\begin{aligned}
& \left| \int_0^t \int_{\Omega} \int_V \left(\int_V |k(x, v, v') \partial_t u(x, v', t)| \, dv' \right) \partial_t u(x, v, t) \, dv \, dx \, dt \right| \\
& \leq C \int_0^t \int_{\Omega} \left(\int_V \left(\int_V |\partial_t u(x, v', t)| \, dv' \right) |\partial_t u(x, v, t)| \, dv \right) \, dx \, dt \\
& \leq C \int_0^t \int_{\Omega} \left(\left(\int_V |\partial_t u(x, v', t)|^2 \, dv' \right)^{1/2} |V|^{1/2} \right) \\
& \quad \times \left(\left(\int_V |\partial_t u(x, v, t)|^2 \, dv \right)^{1/2} |V|^{1/2} \right) \, dx \, dt \\
& = C |V| \int_0^t \int_{\Omega} \int_V |\partial_t u(x, v, t)|^2 \, dv \, dx \, dt,
\end{aligned}$$

where we set $|V| = \int_V dv$. By $R(\cdot, \cdot, 0) \in L^\infty(\Omega \times V)$ by (1.15), using

$$\partial_t u(x, v, 0) = -v \cdot \nabla a - \sigma_t a + \int_V k(x, v, v') a(x, v') \, dv' + fR(x, v, 0),$$

we have

$$E(0) \leq C(\|f\|_{L^2(\Omega \times V)}^2 + \|a\|_{L^2(\Omega \times V)}^2 + \|\nabla a\|_{L^2(\Omega \times V)}^2).$$

Hence, since $\int_0^t \int_{\Gamma_+} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt \geq 0$, by (2.3) we have

$$\begin{aligned}
E(t) & \leq E(0) - \int_0^t \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt + C \|f\|_{L^2(\Omega \times V)}^2 + C \int_0^t E(\eta) \, d\eta \\
& \leq - \int_0^t \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt + C(\|f\|_{L^2(\Omega \times V)}^2 + \|a\|_{L^2(\Omega \times V)}^2 + \|\nabla a\|_{L^2(\Omega \times V)}^2) \\
& \quad + C \int_0^t E(\eta) \, d\eta, \quad 0 \leq t \leq T.
\end{aligned}$$

The Gronwall inequality implies

$$E(t) \leq C \left(- \int_0^t \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt + \|f\|_{L^2(\Omega \times V)}^2 + \|a\|_{L^2(\Omega \times V)}^2 + \|\nabla a\|_{L^2(\Omega \times V)}^2 \right) \tag{2.4}$$

for $0 \leq t \leq T$. Thus, (2.1) is verified.

By (2.3), we have

$$\begin{aligned}
E(T) & \leq E(0) - \int_0^T \int_{\Gamma_+} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt \\
& \quad - \int_0^T \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt + C \int_0^T E(\eta) \, d\eta + C \|f\|_{L^2(\Omega \times V)}^2.
\end{aligned}$$

Hence, (2.4) yields

$$\begin{aligned} & \int_0^T \int_{\Gamma_+} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt \\ & \leq E(0) - E(T) - \int_0^T \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt + C \int_0^T E(\eta) \, d\eta + C \|f\|_{L^2(\Omega \times V)}^2 \\ & \leq -C \int_0^T \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt + C (\|f\|_{L^2(\Omega \times V)}^2 + \|a\|_{L^2(\Omega \times V)}^2 + \|\nabla a\|_{L^2(\Omega \times V)}^2). \end{aligned}$$

Thus, the proof of lemma 2.1 is completed. \square

3. Carleman estimate

In this section, we prove our key Carleman estimate.

We set

$$Q = \Omega \times (0, T)$$

and

$$Pu(x, v, t) = \partial_t u(x, v, t) + v \cdot \nabla u(x, v, t) + \sigma_t(x, v)u(x, v, t), \quad (x, t) \in Q, \quad v \in V.$$

By the assumption (1.8) on V , we can choose $\gamma \in \mathbb{R}^n$ satisfying $(\gamma \cdot v) > 0$ for all $v \in \bar{V}$. We set

$$\varphi(x, t) = -\beta t + (\gamma \cdot x)$$

where $0 < \beta < \min_{v \in \bar{V}} (\gamma \cdot v)$ and

$$B := \partial_t \varphi + (v \cdot \nabla \varphi) = -\beta + (\gamma \cdot v) > 0.$$

Then we can prove

Lemma 3.1. *There exist constants $s_0 > 0$ and $C > 0$, such that*

$$\begin{aligned} & s \int_{\Omega} \int_V |u(x, v, 0)|^2 e^{2s\varphi(x,0)} \, dv \, dx + s^2 \int_Q \int_V |u(x, v, t)|^2 e^{2s\varphi(x,t)} \, dv \, dx \, dt \\ & \leq C \int_Q \int_V |Pu|^2 e^{2s\varphi(x,t)} \, dv \, dx \, dt + s \int_0^T \int_{\Gamma_+} (v \cdot \nu) |u|^2 e^{2s\varphi(x,t)} \, dS \, dv \, dt \end{aligned}$$

for all $s \geq s_0$ and $u \in H^1(0, T; L^2(\Omega \times V))$ satisfying $\nabla u \in L^2(\Omega \times V \times (0, T))$ and $u(\cdot, \cdot, T) = 0$ in $\Omega \times V$.

Remark 3.1. The proof is direct by integration by parts. Lemma 3.1 gives what is called a Carleman estimate with a special choice of linear φ which is possible by the geometric condition (1.8). This Carleman estimate is essential for our proof, and thanks to it we need not extend solutions to $t < 0$. In particular, we note that $u(x, v, 0)$ and $u(x, v, t)$ with the power s^2 are estimated by the right-hand side.

Here and henceforth, $C > 0$ denotes generic constants which are independent of $s > 0$.

Proof. Since $\sigma_t \in L^\infty(\Omega \times V)$, by choosing $s > 0$ large, it suffices to prove the inequality for $\sigma_t = 0$. We set $w(x, v, t) = e^{s\varphi(x,t)} u(x, v, t)$ and $Lw(x, v, t) = e^{s\varphi(x,t)} P(e^{-s\varphi(x,t)} w(x, v, t))$. Henceforth, we omit the independent variables (x, v, t) if there is no fear of confusion. Then

$$Lw = \{\partial_t w + (v \cdot \nabla w)\} - sBw.$$

We have

$$\int_Q |Pu|^2 e^{2s\varphi(x,t)} dx dt = \int_Q |Lw|^2 dx dt.$$

Now, for almost all $v \in V$, by $u(\cdot, v, T) = 0$, we calculate and estimate:

$$\begin{aligned} \int_Q |Pu|^2 e^{2s\varphi} dx dt &= \int_Q |\partial_t w + (v \cdot \nabla w)|^2 dx dt + \int_Q |sB|^2 w^2 dx dt \\ &\quad - 2s \int_Q Bw(\partial_t w + (v \cdot \nabla w)) dx dt \\ &\geq -2s \int_Q Bw(\partial_t w + v \cdot \nabla w) dx dt + s^2 \int_Q B^2 w^2 dx dt \\ &= -s \int_Q (B\partial_t(w^2) + Bv \cdot \nabla(w^2)) dx dt + s^2 \int_Q B^2 w^2 dx dt \\ &= s \int_\Omega B|w(x, 0)|^2 dx - s \int_0^T \int_{\partial\Omega} B(v \cdot \nu)w^2 dS dt + s^2 \int_Q B^2 w^2 dx dt \\ &\geq s \int_\Omega B|w(x, 0)|^2 dx - s \int_0^T \int_{\partial\Omega \cap \{(v \cdot \nu(x)) > 0\}} B(v \cdot \nu)w^2 dS dt \\ &\quad + s^2 \int_Q B^2 w^2 dx dt. \end{aligned}$$

Substituting $w = e^{s\varphi}u$ and noting $B > 0$, we have

$$\begin{aligned} \int_\Omega s|u(x, v, 0)|^2 e^{2s\varphi(x,0)} dx + s^2 \int_Q |u(x, v, t)|^2 e^{2s\varphi} dx dt \\ - s \int_0^T \int_{\partial\Omega \cap \{(v \cdot \nu(x)) > 0\}} (v \cdot \nu)|u(x, v, t)|^2 e^{2s\varphi} dS dt \\ \leq C \int_Q |Pu(x, v, t)|^2 e^{2s\varphi(x,t)} dx dt. \end{aligned}$$

Integrating in v over V , we complete the proof. \square

Finally, we prove a Carleman estimate for the transport equation with the integral term $\int_V ku dv'$. For it, it is essential that $\varphi(x, t) = -\beta t + (\gamma \cdot x)$ is independent of v .

Lemma 3.2. *There exist constants $s_0 > 0$ and $C > 0$, such that*

$$\begin{aligned} s \int_\Omega \int_V |u(x, v, 0)|^2 e^{2s\varphi(x,0)} dv dx + s^2 \int_Q \int_V |u(x, v, t)|^2 e^{2s\varphi} dv dx dt \\ \leq C \int_Q \int_V \left| \partial_t u + v \cdot \nabla u + \sigma_t u - \int_V ku dv' \right|^2 e^{2s\varphi(x,t)} dv dx dt \\ + s \int_0^T \int_{\Gamma_+} (v \cdot \nu)|u|^2 e^{2s\varphi(x,t)} dS dv dt \end{aligned}$$

for all $s \geq s_0$ and $u \in H^1(0, T; L^2(\Omega \times V))$ satisfying $\nabla u \in L^2(\Omega \times V \times (0, T))$ and $u(\cdot, \cdot, T) = 0$ in $\Omega \times V$.

Proof. By $k \in L^\infty(\Omega \times V \times V)$, noting that φ is independent of v , we have

$$\begin{aligned} & \int_Q \int_V \left| \int_V k(x, v, v') u(x, v', t) dv' \right|^2 e^{2s\varphi} dv dx dt \\ & \leq C \int_Q \int_V \left(\int_V |u(x, v', t)|^2 dv' \right) e^{2s\varphi(x,t)} dv dx dt \\ & \leq C|V| \int_Q \int_V |u(x, v', t)|^2 e^{2s\varphi(x,t)} dv' dx dt. \end{aligned}$$

Therefore, we can absorb the term $\int_Q \int_V \left| \int_V k u dv' \right|^2 e^{2s\varphi} dv dx dt$ into the left-hand side by choosing $s > 0$ large. Thus, the lemma follows from lemma 3.1. \square

4. Proof of theorem 1.3

The proof of theorem 1.3 is based on the Carleman estimate (lemma 3.2) and the energy estimate (lemma 2.1), and is similar to [22]. Henceforth, $C > 0$ and C_1 denote generic constants which are independent of $s > 0$. Let $\varphi(x, t) = -\beta t + (\gamma \cdot x)$ for $(x, t) \in Q$. We set

$$r_{\max} = \max_{x \in \bar{\Omega}} (\gamma \cdot x), \quad r_{\min} = \min_{x \in \bar{\Omega}} (\gamma \cdot x).$$

By conditions (1.18) on $T > 0$, we can choose $\beta > 0$, such that

$$0 < \beta < \min_{v \in \bar{V}} (\gamma \cdot v), \quad r_{\max} - \beta T < r_{\min}. \tag{4.1}$$

Then

$$\varphi(x, T) \leq r_{\max} - \beta T < r_{\min} \leq \varphi(x, 0), \quad x \in \bar{\Omega}.$$

Therefore, there exist $\delta > 0$ and r_0, r_1 , such that $r_{\max} - \beta T < r_0 < r_1 < r_{\min}$,

$$\varphi(x, t) > r_1, \quad (x, t) \in \bar{\Omega}, \quad 0 \leq t \leq \delta \tag{4.2}$$

and

$$\varphi(x, t) < r_0, \quad (x, t) \in \bar{\Omega}, \quad T - 2\delta \leq t \leq T. \tag{4.3}$$

For applying lemma 3.2, we need a cut-off function $\chi \in C_0^\infty(\mathbb{R})$, such that $0 \leq \chi \leq 1$ and

$$\chi(t) = \begin{cases} 1, & 0 \leq t \leq T - 2\delta, \\ 0, & T - \delta \leq t \leq T. \end{cases} \tag{4.4}$$

We set

$$z(x, v, t) = (\partial_t u(x, v, t)) \chi(t).$$

Then we have $z(x, v, T) = 0$ and

$$Pz - \int_V k(x, v, v') z dv' = \chi f(\partial_t R) + (\partial_t \chi) \partial_t u, \quad (x, t) \in Q, \quad v \in V,$$

$$\begin{aligned} z(x, v, 0) &= f(x, v) R(x, v, 0) - v \cdot \nabla a(x, v) - \sigma_t a \\ &+ \int_V k(x, v, v') a(x, v') dv', \quad x \in \Omega, \quad v \in V. \end{aligned}$$

Applying lemma 3.2 to z , we obtain

$$\begin{aligned} s \int_\Omega \int_V |z(x, v, 0)|^2 e^{2s\varphi(x,0)} dv dx &\leq C \int_Q \int_V |\chi f \partial_t R|^2 e^{2s\varphi(x,t)} dv dx dt \\ &+ C \int_Q \int_V |(\partial_t \chi) \partial_t u|^2 e^{2s\varphi(x,t)} dv dx dt + C e^{C_1 s} d_0^2. \end{aligned} \tag{4.5}$$

Here, we set

$$d_0 = \left(\int_0^T \int_{\Gamma_+} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt \right)^{1/2}$$

and

$$d = \left(\int_0^T \int_{\Gamma_+} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt \right)^{1/2} + \|a\|_{L^2(\Omega \times V)} + \|\nabla a\|_{L^2(\Omega \times V)},$$

and we used $s e^{Cs} \leq e^{(C+1)s}$ for $t > 0$, so that we replace C in the last term on the right-hand side of (4.5) by $C + 1$, and we set $C_1 = C + 1$. Since $\partial_t \chi = 0$ for $0 \leq t \leq T - 2\delta$ or $T - \delta \leq t \leq T$, by (4.3) we have

$$\begin{aligned} \int_Q \int_V |(\partial_t \chi) \partial_t u|^2 e^{2s\varphi(x,t)} \, dv \, dx \, dt &= \int_{T-2\delta}^{T-\delta} \int_{\Omega} \int_V |(\partial_t \chi) \partial_t u|^2 e^{2s\varphi(x,t)} \, dv \, dx \, dt \\ &\leq C e^{2sr_0} \int_{T-2\delta}^{T-\delta} \int_{\Omega} \int_V |\partial_t u|^2 \, dv \, dx \, dt. \end{aligned} \quad (4.6)$$

By (2.1), we obtain

$$\begin{aligned} \int_{\Omega} \int_V |\partial_t u(x, v, t)|^2 \, dv \, dx &\leq C (\|f\|_{L^2(\Omega \times V)}^2 + \|a\|_{L^2(\Omega \times V)}^2 + \|\nabla a\|_{L^2(\Omega \times V)}^2) \\ &\quad + C \int_0^T \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt \end{aligned}$$

for $0 \leq t \leq T$. Therefore, by (4.6) we obtain

$$\begin{aligned} \int_Q \int_V |(\partial_t \chi) \partial_t u|^2 e^{2s\varphi(x,t)} \, dv \, dx \, dt &\leq C e^{2sr_0} (\|f\|_{L^2(\Omega \times V)}^2 + \|a\|_{L^2(\Omega \times V)}^2 + \|\nabla a\|_{L^2(\Omega \times V)}^2) \\ &\quad - C e^{2sr_0} \int_0^T \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt. \end{aligned}$$

Moreover, since $R(x, v, 0) \neq 0$ and

$$f(x, v)R(x, v, 0) = z(x, v, 0) + v \cdot \nabla a + \sigma_t a - \int_V k(x, v, v') a(x, v') \, dv'$$

for $x \in \bar{\Omega}$ and $v \in \bar{V}$, we have

$$\begin{aligned} \int_{\Omega} \int_V |z(x, v, 0)|^2 e^{2s\varphi(x,0)} \, dv \, dx &+ C e^{C_1 s} (\|a\|_{L^2(\Omega \times V)}^2 + \|\nabla a\|_{L^2(\Omega \times V)}^2) \\ &\geq C \int_{\Omega} \int_V |f(x, v)|^2 e^{2s\varphi(x,0)} \, dv \, dx. \end{aligned}$$

Therefore, (4.5) yields

$$\begin{aligned} s \int_{\Omega} \int_V |f(x, v)|^2 e^{2s\varphi(x,0)} \, dv \, dx &\leq C \int_Q \int_V |f(x, v)|^2 e^{2s\varphi(x,t)} \, dv \, dx \, dt \\ &\quad + C e^{2sr_0} \|f\|_{L^2(\Omega \times V)}^2 - C e^{2sr_0} \int_0^T \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt + C e^{C_1 s} d^2. \end{aligned}$$

Since $\varphi(x, t) \leq \varphi(x, 0)$ for $(x, t) \in Q$, we have

$$\begin{aligned} s \int_{\Omega} \int_V |f(x, v)|^2 e^{2s\varphi(x,0)} \, dv \, dx &\leq C \int_0^T \int_{\Omega} \int_V |f(x, v)|^2 e^{2s\varphi(x,0)} \, dv \, dx \, dt \\ &\quad + C e^{2sr_0} \|f\|_{L^2(\Omega \times V)}^2 - C e^{2sr_0} \int_0^T \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 \, dS \, dv \, dt + C e^{C_1 s} d^2. \end{aligned}$$

That is,

$$(s - CT) \int_{\Omega} \int_V |f(x, v)|^2 e^{2s\varphi(x,0)} dv dx \leq C e^{2sr_0} \|f\|_{L^2(\Omega \times V)}^2 - C e^{2sr_0} \int_0^T \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 dS dv dt + C e^{C_1 s} d^2$$

for all large $s > 0$. Using $\varphi(x, 0) > r_1$ and choosing $s > 0$ large, we obtain

$$s e^{2sr_1} \int_{\Omega} \int_V |f(x, v)|^2 dv dx \leq C e^{2sr_0} \|f\|_{L^2(\Omega \times V)}^2 - C e^{C_1 s} \int_0^T \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 dS dv dt + C e^{C_1 s} d^2.$$

That is,

$$\|f\|_{L^2(\Omega \times V)}^2 \leq C e^{-2s(r_1 - r_0)} \|f\|_{L^2(\Omega \times V)}^2 - C e^{C_1 s} \int_0^T \int_{\Gamma_-} (v \cdot \nu) |\partial_t u|^2 dS dv dt + C e^{C_1 s} (\|a\|_{L^2(\Omega \times V)}^2 + \|\nabla a\|_{L^2(\Omega \times V)}^2).$$

for all large $s > 0$. Note that $r_1 - r_0 > 0$. Choosing $s > 0$ large, we can absorb the first term on the right-hand side into the left-hand side, and complete the proof. \square

Remark 4.1. If we assume $\|\partial_t u\|_{L^2(\Omega \times V \times (0, T))} \leq M$ with some constant $M > 0$, the estimate in (4.6) is written as

$$\int_Q \int_V |(\partial_t \chi) \partial_t u|^2 e^{2s\varphi(x,t)} dv dx dt \leq C e^{2sr_0} M^2.$$

Then f is estimated less sharply but more easily without using (2.1). We obtain

$$\|f\|_{L^2(\Omega \times V)}^2 \leq CM^2 e^{-2s(r_1 - r_0)} + C_1 e^{C_1 s} d^2.$$

By minimizing the right-hand side with respect to s , the Hölder stability is obtained only with data on the sub-boundary $\Gamma_+ \times (0, T)$. That is, there exist constants $\kappa \in (0, 1)$, $C > 0$ and $T > 0$, such that

$$\|f\|_{L^2(\Omega \times V)} \leq C \left(\left(\int_0^T \int_{\Gamma_+} (v \cdot \nu) |\partial_t u|^2 dv dS dt \right)^{1/2} + \|a\|_{L^2(\Omega \times V)} + \|\nabla a\|_{L^2(\Omega \times V)} \right)^\kappa$$

for all $f \in L^2(\Omega \times V)$. The data on the whole boundary $V \times \partial\Omega \times (0, T)$ are needed to obtain the Lipschitz stability.

5. Concluding remarks

Considering the transport equation (1.9) over $0 < t < T$, we establish the Lipschitz stability in determining σ_t or σ_s by measurement data on the sub-boundary $\Gamma_+ \times (0, T)$ by choosing the initial value a and boundary value g on $\Gamma_- \times (0, T)$.

We consider the inverse problem totally in $(0, T)$. For a conventional method by Bukgheim and Klivanov [10], it is necessary to extend the solution $u(x, v, t)$ to $-T < t < 0$ and such an extension needs some extra condition on σ_t (see [30]). If we consider the transport equation in $t \in (-T, T)$ with the ‘intermediate’ value $u(x, v, 0)$ at $t = 0$, then we need not the extension of u in t and we can more directly prove the Lipschitz stability (see [29]). However, in the latter case, since we have to control $u(x, v, 0)$ in order to satisfy the positivity (1.12), the control is more difficult, because it is not an initial condition.

We have to assume an extra condition (1.8) on V . It is not known whether we can prove theorems 1.1–1.4 for general V satisfying only $0 \notin \bar{V}$.

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References

- [1] Arridge S R 1999 Optical tomography in medical imaging *Inverse Problems* **15** R41–R93
- [2] Arridge S R and Schotland J C 2009 Optical tomography: forward and inverse problems *Inverse Problems* **25** 123010
- [3] Bal G 2009 Inverse transport theory and applications *Inverse Problems* **25** 053001
- [4] Bal G, Langmore I and Monard F 2008 Inverse transport with isotropic sources and angularly averaged measurements *Inverse Problems Imaging* **2** 23–42
- [5] Bal G and Jollivet A 2008 Stability estimates in stationary inverse transport *Inverse Problems Imaging* **2** 427–54
- [6] Bal G and Jollivet A 2009 Time-dependent angularly averaged inverse transport *Inverse Problems* **25** 075010
- [7] Bal G and Jollivet A 2010 Stability for time-dependent inverse transport *SIAM J. Math. Anal.* **42** 679–700
- [8] Bardos C 1970 Problèmes aux limites pour les équations aux dérivées partielles du premier ordre à coefficients réels; théorèmes d’approximation; applications à l’équations de transport *Ann. Sci. Ecole Norm Sup.* **3** 185–233
- [9] Beilina L and Klibanov M V 2012 *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems* (New York: Springer)
- [10] Bukhgeim A L and Klibanov M V 1981 Global uniqueness of a class of multidimensional inverse problems *Sov. Math.—Dokl.* **24** 244–7
- [11] Carleman T 1939 Sur un problème d’unicité pour les systèmes d’équations aux dérivées partielles à deux variables indépendantes *Ark. Mat. Astr. Fys.* **B 26** 1–9
- [12] Case K M and Zweifel P F 1967 *Linear Transport Theory* (Boston, MA: Addison-Wesley)
- [13] Chandrasekhar S 1960 *Radiative Transfer* (New York: Dover)
- [14] Cipolatti R and Yamamoto M 2011 An inverse problem for a wave equation with arbitrary initial values and a finite time of observations *Inverse Problems* **27** 095006
- [15] Choulli M and Stefanov P 1996 Inverse scattering and inverse boundary value problems for the linear Boltzmann equation *Commun. Partial Differ. Eqns* **21** 763–85
- [16] Choulli M and Stefanov P 1999 An inverse boundary value problem for the stationary transport equation *Osaka J. Math.* **36** 87–104
- [17] Douglis A 1966 The solutions of multidimensional generalized transport equations and their calculation by difference methods *Numerical Solution of Partial Differential Equations* (New York: Academic) pp 197–256
- [18] Duderstadt J J and Martin W R 1979 *Transport Theory* (New York: Wiley)
- [19] Gaitan P and Ouzzane H 2013 Inverse problem for a free transport equation using Carleman estimates *Appl. Anal.* **at press**
- [20] Hörmander L 1963 *Linear Partial Differential Operators* (Berlin: Springer)
- [21] Imanuvilov O and Yamamoto M 2001 Global uniqueness and stability in determining coefficients of wave equations *Commun. Partial Differ. Eqns* **26** 1409–25
- [22] Imanuvilov O and Yamamoto M 2001 Global Lipschitz stability in an inverse hyperbolic problem by interior observations *Inverse Problems* **17** 717–28
- [23] Isakov V 1990 *Inverse Source Problems* (Providence, RI: American Mathematical Society)
- [24] Isakov V 1993 Carleman type estimates in an anisotropic case and applications *J. Diff. Eqns* **105** 217–38
- [25] Isakov V 2006 *Inverse Problems for Partial Differential Equations* (Berlin: Springer)
- [26] Klibanov M V 1984 Inverse problems in the ‘large’ and Carleman bounds *Differ. Eqns* **20** 755–60
- [27] Klibanov M V 1992 Inverse problems and Carleman estimates *Inverse Problems* **8** 575–96
- [28] Klibanov M V 2013 Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems *J. Inverse Ill-posed Problems* **21** 477–560
- [29] Klibanov M V and Pamyatnykh S E 2006 Lipschitz stability of a non-standard problem for the non-stationary transport equation via a Carleman estimate *Inverse Problems* **22** 881–90

- [30] Klibanov M V and Pamyatnykh S E 2008 Global uniqueness for a coefficient inverse problem for the non-stationary transport equation via Carleman estimate *J. Math. Anal. Appl.* **343** 352–65
- [31] Klibanov M V and Timonov A 2004 *Carleman Estimates for Coefficient Inverse Problems and Numerical Applications* (Utrecht: VSP)
- [32] Klibanov M V and Yamamoto M 2007 Exact controllability for the time dependent transport equation *SIAM J. Control Optim.* **46** 2071–195
- [33] Lavrent'ev M M, Romanov V G and Shishat-skiĭ S P 1986 *Ill-posed Problems of Mathematical Physics and Analysis* (Providence, RI: American Mathematical Society)
- [34] McDowall S, Stefanov P and Tamasan A 2010 Stability of the gauge equivalent classes in inverse stationary transport *Inverse Problems* **26** 025006
- [35] Prilepko A I and Ivankov A L 1984 Inverse problems for the time-dependent transport equation *Sov. Math.—Dokl.* **29** 559–64
- [36] Romanov V G 1997 Stability estimates in the three-dimensional inverse problem for the transport equation *J. Inverse Ill-posed Problems* **5** 463–75
- [37] Romanov V G 1998 A conditional stability theorem in the problem of determining the dispersion index and relaxation for the stationary transport equation *Mat. Tr.* **1** 78–115 (Engl. transl.)
- [38] Sobolev V V 1975 *Light Scattering in Planetary Atmospheres* (Oxford: Pergamon)
- [39] Stefanov P 2003 Inverse problems in transport theory *Inside Out: Inverse Problems and Applications* ed G Uhlmann (Cambridge: Cambridge University Press) pp 111–31
- [40] Stefanov P and Tamasan A 2009 Uniqueness and non-uniqueness in inverse radiative transfer *Proc. Am. Math. Soc.* **137** 2335–44
- [41] Stefanov P and Uhlmann G 2003 Optical tomography in two dimensions *Methods Appl. Anal.* **10** 1–10
- [42] Tamasan A 2002 An inverse boundary value problem in two-dimensional transport *Inverse Problems* **18** 209–19
- [43] Ukai S 1976 *Transport Equations* (Tokyo: Sangyo-tosyo) (in Japanese)
- [44] Ukai S 1986 Solutions of Boltzmann equation *Patterns and Waves* (Amsterdam: Elsevier) pp 37–96
- [45] Wang J-N 1999 Stability estimates of an inverse problem for the stationary transport equation *Ann. Inst. Henri Poincaré A* **70** 473–95
- [46] Yamamoto M 2009 Carleman estimates for parabolic equations and applications *Inverse Problems* **25** 123013
- [47] Yuan G and Yamamoto M 2009 Lipschitz stability in the determination of the principal part of a parabolic equation *ESAIM Control Optim. Calc. Var.* **15** 525–54