

## Frequency Dependence of Quantum Localization in a Periodically Driven System

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We study the quantum localization of a GOE random matrix model driven by a periodic external field. After the diffusion is suppressed, the average energy oscillates around some finite value  $E_{\text{sat}}$ . We study how  $E_{\text{sat}}$  depends on the frequency of the external field. In order to investigate this phenomenon, we also study the frequency dependence of the relevant number of Floquet states  $N_{\text{min}}$ , and discuss its dependence in the view of the Anderson localization in the region where the frequency is not large.

Recently active research has been done on the dynamical nature of quantum systems whose energy levels have complex structures. Especially, the quantum localization is known to take place in these systems under a periodic external field. The quantum localization was first predicted theoretically<sup>1)</sup> and also realized experimentally.<sup>2)-6)</sup> Although the quantum localization has mainly studied for simple models such as the kicked rotator model<sup>1)</sup> and the kicked top model,<sup>7)</sup> some studies have recently done on random matrix models which take account of many-body interaction.<sup>8),9)</sup> In this paper, we investigate the frequency dependence of the quantum localization of the system of a random matrix model.

We consider time-reversal invariant systems under an oscillating external field. The random matrix ensemble appropriate for describing the spectral statistics of these systems is Gaussian orthogonal ensemble (GOE).<sup>10)</sup> Eigenvalues of a GOE random matrix repulse each other, and they never degenerate. Points where energy levels nearly degenerate are called avoided crossings. We take  $H_0$  and  $V$  to be GOE random matrices. Elements of these matrices are taken from the Gaussian distribution. The mean value of each element is zero, and the variance is 1 for diagonal element and 2 for off-diagonal element. Using  $H_0$  and  $V$ , the Hamiltonian is given as follows;

$$\mathcal{H}(t) = H_0 + \lambda(t)V \quad (1)$$

with  $\lambda(t) = A \sin(\omega t)$  ( $A = 0.5$ ).

We take the ground state as the initial state  $|\psi_0\rangle$  at time  $t = 0$ . State  $|\psi_1\rangle$  after a period is expressed by operating the time-evolution operator for a period  $F$  (the Floquet operator<sup>11)</sup>);  $|\psi_1\rangle = F|\psi_0\rangle$ . Eigenfunctions of  $F$  form the complete set of the Hilbert space and eigenvalues of  $F$  are aligned on a unit circle in the complex plane;

$$F|\nu\rangle = e^{i\phi_\nu}|\nu\rangle. \quad (2)$$

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In the same way, the state after the  $n$ th period is expressed as,  $|\psi_n\rangle = F^n|\psi_0\rangle$ . Therefore, the average energy after the  $n$ th period is written as,

$$\langle\psi_n|H_0|\psi_n\rangle = \sum_{\nu',\nu} e^{in(\phi_{\nu'}-\phi_{\nu})} \langle\psi_0|\nu'\rangle \langle\nu|\psi_0\rangle \langle\nu'|H_0|\nu\rangle. \quad (3)$$

Note that  $\mathcal{H}(t=0) = H_0$ .

In this equation, the energy oscillates around

$$E_{\text{sat}} \equiv \sum_{\nu} |\langle\nu|\psi_0\rangle|^2 \langle\nu|H_0|\nu\rangle, \quad (4)$$

after the diffusion is suppressed. Hence, we regard  $E_{\text{sat}}$  as the saturated energy. In the present study, we normalize  $E_{\text{sat}}$  such that the distance between  $E = 0$  and the ground state is 1. Figure 1 shows  $E_{\text{sat}}$  as a function of  $\omega$  for systems of several sizes. When  $\omega$  is very small, the average energy hardly changes due to the adiabatic theorem:<sup>12)</sup> when  $\omega$  gets slightly larger, transfers of occupation probabilities of energy levels begin to occur at avoided crossings. We see that  $E_{\text{sat}}$  approaches to the center of the energy spectrum ( $E_{\text{sat}} = 0$ ) asymptotically as  $\omega$  grows. Note that the energy spectrum is symmetric about its center.

As is seen in Eq. (4), most of the  $\omega$ -dependence of  $E_{\text{sat}}$  is determined by  $|\langle\nu|\psi_0\rangle|^2$  because Eq. (4) is approximately rewritten as  $E_{\text{sat}} \simeq \sum_{\nu} |\langle\nu|\psi_0\rangle|^2 E_{\nu}$ , where  $E_{\nu}$  is eigenvalue of  $H_0$  whose eigenvector is the closest to  $|\nu\rangle$ . Therefore we focus on this overlap and employ the minimal number  $N_{\text{min}}$  of Floquet eigenvectors necessary to cover the initial state  $|\psi_0\rangle$  to within a ratio  $r$ .<sup>7)</sup> That is, we define  $N_{\text{min}}$  as follows;

$$N_{\text{min}} = \frac{1}{N} \min \left\{ \mathcal{N}_{\text{min}} : \sum_{\nu=1}^{\mathcal{N}_{\text{min}}} |\langle\psi_0|\nu\rangle|^2 > r; |\langle\psi_0|\nu\rangle|^2 \geq |\langle\psi_0|\nu+1\rangle|^2 \right\}, \quad (5)$$

where  $N$  is the size of the Hamiltonian. We take  $r = 0.99$ . Figure 2 shows  $N_{\text{min}}$  as a function of  $\omega$  for  $N = 256, 500$  and  $700$ . In Fig. 2, when  $\omega$  is nearly zero,  $N_{\text{min}}$

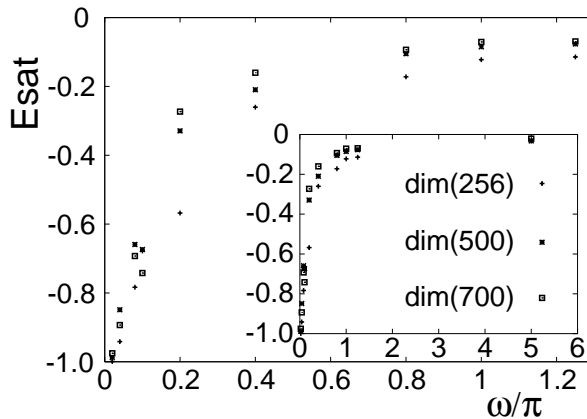


Fig. 1.  $E_{\text{sat}}$  as a function of  $\omega$ . The  $E_{\text{sat}}$  for 256, 500 and 700 dimension are shown respectively. The inset also shows  $E_{\text{sat}}$  as a function of  $\omega$  for a wide range of  $\omega$ .

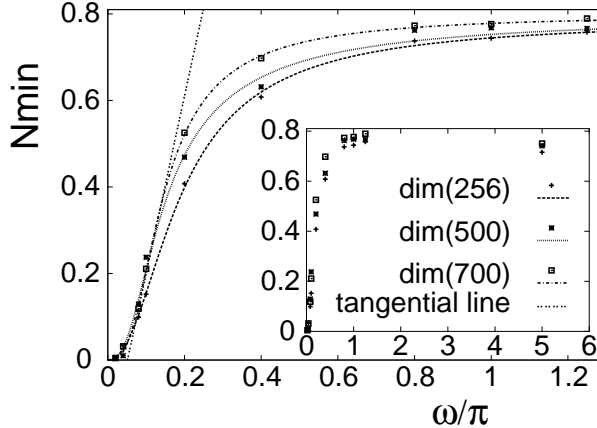


Fig. 2.  $N_{\min}$  as a function of  $\omega$ . The  $N_{\min}$  for 256, 500 and 700 dimension are shown respectively. The inset also shows  $N_{\min}$  as a function of  $\omega$  for a wide range of  $\omega$ .

is close to zero because almost no transitions of states occur due to the adiabatic theorem.<sup>12)</sup> On the other hand,  $N_{\min}$  is approximately constant when  $\omega$  is sufficiently large because the occupation probability is spread over almost the whole states before the quantum diffusion is suppressed.

In Fig. 2, we also find empirically that the form of  $N_{\min}(\omega)$  is fitted by

$$N_{\min}(\omega) = N_{\min}^* \left\{ 1 - \frac{1}{[(\omega/\omega^*)^a + 1]^b} \right\}. \quad (6)$$

Table I. Values of the parameters in Eq. (6).

They are obtained by the least-squares method.

parameter	dim(256)	dim(500)	dim(700)
$N_{\min}^*$	0.79	0.80	0.80
$a$	2.3	2.6	3.0
$b$	0.67	0.47	0.54
$\omega^*$	0.15	0.10	0.11

The values of  $a$ ,  $b$  and  $\omega^*$  obtained from the least-squares method are shown in Table I. In Eq. (6),  $N_{\min}(\omega)$  has a linear shape in the intermediate region around the inflection point ( $\omega \sim 0.1\pi$ ). Hence, we regard the tangential line at the inflection point as  $N_{\min}(\omega)$  in the intermediate region. The tangential line at the inflection point of  $N_{\min}$  in Eq. (6) in the case  $N = 700$  is also depicted in Fig. 2.

Now, we focus on the intermediate region, where  $N_{\min}$  grows linearly. We find numerically that transitions of states take place only at avoided level crossings when  $\omega \sim 0.1\pi$ . We investigate the quantum localization in the following way, since a direct application of the Landau-Zener transition<sup>13)</sup> does not explain the observed  $\omega$ -dependence of  $E_{\text{sat}}$  and  $N_{\min}$ .<sup>14), 15)</sup> For the localization, the effect of quantum interference takes important role. In the following, we associate the quantum localization of the system with the Anderson localization.<sup>16)</sup> Here we study the case with small  $\omega$ , where the Landau-Zener transition gives main mechanism of level transfer.

The corresponding system, which shows the Anderson localization, has random potential  $U$  with the width  $W$  and constant hopping  $t$ . Since transitions occur at

avoided crossings, we believe that the Landau-Zener transition takes place and the transition probability at each avoided crossing is proportional to  $\exp\left(-\frac{\pi\epsilon^2}{2\sigma A\omega \cos\omega t}\right)$ , where  $\epsilon$  is the size of gaps of avoided crossings, and  $\sigma$  is the size of the difference of slopes of two asymptotic lines at avoided crossings. We assume that the corresponding hopping constant  $t$  after a period of the external field has the form

$$t \sim \exp\left[-h(N, A)\frac{\bar{\epsilon}^2}{\bar{\sigma}}\omega^{-1}\right], \quad (7)$$

where  $h(N, A)$  is some function of  $N$  and  $A$ ,  $\bar{\epsilon}$  is the typical size of  $\epsilon$ , and  $\bar{\sigma}$  is the typical size of  $\sigma$ . The probability  $p_m$  that the state occupies the  $m$ th site from the initial 0th site is given in the study of the Anderson localization by<sup>17)</sup>

$$p_m \sim \left(\frac{|t|}{W}\right)^m \sim \exp\left[-h(N, A)\frac{\bar{\epsilon}^2}{\bar{\sigma}}\omega^{-1}m\right]. \quad (8)$$

On the other hand, the probability to find the state at the  $m$ th level of  $\mathcal{H}(t)$  is found to be proportional to  $\exp(-\Gamma m)$ , where  $\Gamma^{-1}$  is the localization length. Comparing them, we have,  $\Gamma \sim h(N, A)\frac{\bar{\epsilon}^2}{\bar{\sigma}}\omega^{-1}$ . Using the fact  $N_{\min} \sim \Gamma^{-1}$ , we obtain

$$N_{\min} \sim \frac{\bar{\sigma}}{\bar{\epsilon}^2}\omega. \quad (9)$$

Equation (9) suggests linear dependence of  $N_{\min}$  on  $\omega$ . In Fig. 2, we find this relation except for very small value of  $\omega$ .

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