Brief Introduction to Case's Method‡

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March 31, 2015

1. Introduction

The radiative transport equation or the linear Boltzmann equation is used, for example, to describe light transport in random media or neutron transport in reactors. The equation can be solved with separation of variables§ if its coefficients are constant. For simplicity let us consider light transport in an infinite medium. Moreover in this note we assume planar symmetry, i.e., the specific intensity $u(x,\mu)$ depends on $x \in \mathbb{R}$ and $\mu \in [-1,1]$, where $\mu = \cos\theta$ with θ the polar angle in three dimensions. Assuming isotropic scattering, the radiative transport equation is given by

$$\mu\frac{\partial u}{\partial x}(x,\mu) + u(x,\mu) = \frac{\varpi}{2}\int_{-1}^{1} u(x,\mu')\,d\mu' + S(x,\mu), \tag{1}$$

where $S(x,\mu)$ is the source term. The parameter $\varpi \in (0,1)$, which is the ratio between the scattering coefficient and the total attenuation, is called the albedo for single scattering. The solution u to (1) is obtained as a linear combination of eigenmodes or separated solutions. This u is the specific intensity of light at position $x \in \mathbb{R}$ and direction $\theta \in [0,\pi]$. The method of solving (1) with separation of variables is called Case's method‖. As we will see, the eigenmodes are singular and given by generalized functions.

2. Eigenvalues and Singular Eigenfunctions

Separated solutions are solutions of the homogeneous radiative transport equation,

$$\mu\frac{\partial u}{\partial x}(x,\mu) + u(x,\mu) = \frac{\varpi}{2}\int_{-1}^{1} u(x,\mu')\,d\mu'. \tag{2}$$

To find solutions to (2), we assume u of the form

$$u_\nu(x,\mu) = \phi_\nu(\mu)e^{-x/\nu}, \tag{3}$$

‡ This note was written for a 1.5-hour lecture in Math 651 (Waves and Imaging in Random Media) at University of Michigan.

§ "Separation of variables is of very limited utility but when it works it is very informative." Jeffrey Rauch (Partial Differential Equations, Springer-Verlag, p. 211)

‖ Kenneth Myron Case was a professor of physics at the University of Michigan between 1951 and 1969. In 1960, he published the paper "Elementary Solutions of the Transport Equation and Their Applications" (Ann. Phys. **9** (1960) 1–23). In 1967, he wrote the textbook "Linear Transport Theory" with Paul F. Zweifel and explained his method, which is now called Case's method. Sometimes the word *Caseology* is used. The method is explained also in the book "Transport Theory" by Duderstadt and William R. Martin.

Case was born on September 23, 1923 in New York, NY and passed away on February 1, 2006. He received his Ph. D. in physics at Harvard University in 1948 under the supervision of J. S. Schwinger (Nobel Prize in physics in 1965 with R. Feynman and S. Tomonaga). After spending some time at Institute for Advanced Study, UC Berkeley, and University of Rochester, he arrived at Ann Arbor as a young assistant professor. Then he became a professor of physics. He left the town in 1969 and became a professor of physics at Rockefeller University.

where ν is a separation constant. We call u_ν eigenmodes. We normalize ϕ_ν as

$$\int_{-1}^{1} \phi_\nu(\mu')\, d\mu' = 1. \tag{4}$$

By substituting u_ν in (3) for u in (2), we obtain

$$(\nu - \mu)\phi_\nu(\mu) = \frac{\varpi\nu}{2}. \tag{5}$$

By carefully looking at (5), we find ϕ_ν as

$$\phi_\nu(\mu) = \frac{\varpi\nu}{2}\mathcal{P}\frac{1}{\nu - \mu} + \lambda(\nu)\delta(\nu - \mu), \tag{6}$$

where

$$\lambda(\nu) = 1 - \frac{\varpi\nu}{2}\mathcal{P}\int_{-1}^{1} \frac{1}{\nu - \mu}\, d\mu.$$

Here $\mathcal{P}$ denotes the Cauchy principal value and $\delta(\cdot)$ is the Dirac delta function. These ϕ_ν are called Case's singular eigenfunctions (Note that they are actually generalized functions). We have

$$\lambda(\nu) = 1 - \varpi\nu \tanh^{-1} \nu.$$

2.1. Discrete Eigenvalues

For $\nu \notin [-1, 1]$, we have

$$\phi_\nu(\mu) = \frac{\varpi\nu}{2}\frac{1}{\nu - \mu}.$$

In this case, ν is obtained as a solution to

$$\Lambda(\nu) = 0. \tag{7}$$

Here $\Lambda(z)$ is defined for $z \in \mathbb{C}$ as

$$\Lambda(z) = 1 - \frac{cz}{2}\int_{-1}^{1} \frac{1}{z - \mu}\, d\mu.$$

We calculate $\Lambda(z)$ as

$$\Lambda(z) = 1 - \frac{\varpi z}{2} \ln\left[\frac{1 + 1/z}{1 - 1/z}\right] = 1 - \varpi z \tanh^{-1}\frac{1}{z}.$$

The function $\Lambda(z)$ has two roots $\pm\nu_0$. In biological tissue, ϖ is close to 1. In such a case, we have

$$\frac{\varpi}{3\nu_0^2} = 1 - \varpi\left[1 + \frac{1}{5\nu_0^4} + \frac{1}{7\nu_0^6} + \cdots\right].$$

Therefore $\nu_0 \simeq 1/\sqrt{3(1 - \varpi)}$ $(0 < 1 - \varpi \ll 1)$. These $\pm\nu_0$ are called discrete eigenvalues.

2.2. Continuous Spectrum

Any ν on $(-1,1)$ labels ϕ_ν in (6). Using the residue theorem, for $\nu \in (-1,1)$ we have

$$\Lambda^{\pm}(\nu) = \lim_{\epsilon\to 0+} \Lambda(\nu \pm i\epsilon) = \lambda(\nu) \pm \frac{i\pi\varpi\nu}{2}. \tag{8}$$

Hence we have

$$\Lambda^{+}(\nu) - \Lambda^{-}(\nu) = i\pi\varpi\nu, \tag{9}$$

$$\lambda(\nu) = \frac{1}{2}\left[\Lambda^{+}(\nu) + \Lambda^{-}(\nu)\right]. \tag{10}$$

3. Orthogonality and Normalization

Theorem 3.1. *(Orthogonality). The functions $\phi_\nu(\mu)$ are orthogonal to each other.*

$$\int_{-1}^{1} \mu\phi_\nu(\mu)\phi_{\nu'}(\mu)\,d\mu = 0, \quad \nu \neq \nu', \tag{11}$$

where ν, ν' are discrete eigenvalues or in the continuous spectrum.

Proof. We multiply (5) by $\phi_{\nu'}(\mu)$.

$$\left(1 - \frac{\mu}{\nu}\right)\phi_\nu(\mu)\phi_{\nu'}(\mu) = \frac{\varpi}{2}\phi_{\nu'}(\mu).$$

Similarly we write (5) with ν' and multiply it by $\phi_\nu(\mu)$.

$$\left(1 - \frac{\mu}{\nu'}\right)\phi_{\nu'}(\mu)\phi_\nu(\mu) = \frac{\varpi}{2}\phi_\nu(\mu).$$

By subtraction we have

$$\left(\frac{\mu}{\nu'} - \frac{\mu}{\nu}\right)\phi_\nu(\mu)\phi_{\nu'}(\mu) = \frac{\varpi}{2}\left[\phi_{\nu'}(\mu) - \phi_\nu(\mu)\right].$$

By integrating over μ and using (4), we find

$$\left(\frac{1}{\nu'} - \frac{1}{\nu}\right)\int_{-1}^{1} \mu\phi_\nu(\mu)\phi_{\nu'}(\mu)\,d\mu = 0.$$

This proves the orthogonality theorem. □

Theorem 3.2. *(Normalization). The functions $\phi_\nu(\mu)$ are orthogonal to each other.*

$$\int_{-1}^{1} \mu\phi_\nu(\mu)\phi_{\nu'}(\mu)\,d\mu = \mathcal{N}(\nu)\delta_{\nu\nu'}, \tag{12}$$

where ν, ν' are discrete eigenvalues or in the continuous spectrum, and $\delta_{\nu\nu'}$ is the Kronecker delta. If both ν, ν' are in the continuous spectrum, $\delta_{\nu\nu'}$ is read as the Dirac delta $\delta(\nu - \nu')$. Here

$$\mathcal{N}(\nu) = \begin{cases} \pm N_0 & \nu = \pm\nu_0, \\ N(\nu) & \nu \in (-1,1), \end{cases}$$

where

$$N_0 = \frac{\varpi\nu_0^2}{2}\frac{d\Lambda(z)}{dz}\bigg|_{z=\nu_0} = \frac{\varpi\nu_0^3}{2}\left(\frac{\varpi}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right),$$

and

$$N(\nu) = \nu\Lambda^{+}(\nu)\Lambda^{-}(\nu) = \nu\left[\left(\frac{\varpi\pi\nu}{2}\right)^2 + \left(1 - \varpi\nu\tanh^{-1}\nu\right)^2\right].$$

Proof. Since we have (11), we will focus on two cases: $\nu, \nu' \notin [-1, 1]$ and $\nu, \nu' \in (-1, 1)$. Consider

$$\int_{-1}^{1} \frac{\mu}{z-\mu}\, d\mu = \int_{-1}^{1} \frac{\mu - z}{z-\mu}\, d\mu + \int_{-1}^{1} \frac{z}{z-\mu}\, d\mu = \frac{2}{\varpi}(1-\varpi) - \frac{2}{\varpi}\Lambda(z).$$

Let us define

$$J(z, z') = \int_{-1}^{1} \frac{1}{z-\mu}\frac{1}{z'-\mu}\mu\, d\mu.$$

We have

$$J(z, z') = \frac{1}{z-z'}\int_{-1}^{1}\left(\frac{1}{z'-\mu} - \frac{1}{z-\mu}\right)\mu\, d\mu = \left(\frac{2}{\varpi}\right)\frac{\Lambda(z)-\Lambda(z')}{z-z'}.$$

We set $z = \pm\nu_0$ and let $z' \to \pm\nu_0$. We obtain

$$\begin{aligned}
\int_{-1}^{1} \mu\phi_{0\pm}^2(\mu)\, d\mu &= \left(\frac{\varpi\nu_0}{2}\right)^2 J(\pm\nu_0, \pm\nu_0) = \pm\frac{\varpi\nu_0^2}{2}\frac{d\Lambda(z)}{dz}\bigg|_{z=\nu_0} \\
&= \mp\frac{\varpi^2\nu_0^2}{2}\frac{d}{dz} z\tanh^{-1}\frac{1}{z}\bigg|_{z=\nu_0} = \pm\frac{\varpi^2\nu_0^2}{2}\left(\frac{\nu_0}{\nu_0^2-1} - \frac{1}{\varpi\nu_0}\right) \\
&= \pm N_0,
\end{aligned}$$

where we used $\Lambda(\nu_0) = 0 \Leftrightarrow \tanh^{-1}(1/\nu_0) = 1/\varpi\nu_0$.

Next we suppose $\nu, \nu' \in (-1, 1)$ and $\nu \neq \nu'$. We have

$$\begin{aligned}
\int_{-1}^{1} \mu\phi_\nu(\mu)\phi_{\nu'}(\mu)\, d\mu = &\left(\frac{\varpi}{2}\right)^2 \nu\nu' \int_{-1}^{1} \frac{\mathcal{P}}{\nu-\mu}\frac{\mathcal{P}}{\nu'-\mu}\mu\, d\mu \\
&+ \frac{\varpi\nu}{2}\lambda(\nu')\int_{-1}^{1}\frac{\mathcal{P}}{\nu-\mu}\delta(\nu'-\mu)\mu\, d\mu \\
&+ \frac{\varpi\nu'}{2}\lambda(\nu)\int_{-1}^{1}\delta(\nu-\mu)\frac{\mathcal{P}}{\nu'-\mu}\mu\, d\mu \\
&+ \lambda(\nu)\lambda(\nu')\int_{-1}^{1}\delta(\nu-\mu)\delta(\nu'-\mu)\mu\, d\mu. \qquad (13)
\end{aligned}$$

We use the Poincaré-Bertrand formula¶ (Appendix C),

$$\frac{\mathcal{P}}{\nu-\mu}\frac{\mathcal{P}}{\nu'-\mu} = \frac{1}{\nu-\nu'}\left(\frac{\mathcal{P}}{\nu'-\mu} - \frac{\mathcal{P}}{\nu-\mu}\right) + \pi^2\delta(\nu-\mu)\delta(\nu'-\mu). \qquad (14)$$

The first term of the right-hand side of (13) is obtained as

$$\begin{aligned}
\text{First term} = &\left(\frac{\varpi}{2}\right)^2\nu\nu'\left[\frac{1}{\nu-\nu'}\left(\mathcal{P}\!\int_{-1}^{1}\frac{\mu}{\nu'-\mu}\, d\mu - \mathcal{P}\!\int_{-1}^{1}\frac{\mu}{\nu-\mu}\, d\mu\right)\right. \\
&\left. + \pi^2\int_{-1}^{1}\delta(\nu-\mu)\delta(\nu'-\mu)\mu\, d\mu\right] \\
= &\left(\frac{\varpi}{2}\right)^2\nu\nu'\left[\frac{1}{\nu-\nu'}\frac{2}{\varpi}\Big(\lambda(\nu)-\lambda(\nu')\Big) + \pi^2\nu\delta(\nu-\nu')\right] \\
= &\frac{\varpi\nu\nu'}{2}\frac{\lambda(\nu)-\lambda(\nu')}{\nu-\nu'} + \left(\frac{\varpi\pi}{2}\right)^2\nu^3\delta(\nu-\nu').
\end{aligned}$$

¶ See for example "Singular Integral Equations" by Muskhelishvili.

The second and third terms of the right-hand side of (13) are calculated as $(\varpi\nu\nu'/2)\lambda(\nu')/(\nu-\nu')$ and $(\varpi\nu\nu'/2)\lambda(\nu)/(\nu'-\nu)$, respectively. By taking the limit $\nu'\to\nu$, we obtain

$$\begin{aligned}\int_{-1}^{1}\mu\phi_\nu(\mu)\phi_{\nu'}(\mu)\,d\mu &= \left[\left(\frac{\varpi\pi\nu}{2}\right)^2+\lambda(\nu)^2\right]\nu\delta(\nu-\nu')\\ &=\nu\Lambda^+(\nu)\Lambda^-(\nu)\delta(\nu-\nu')\\ &=N(\nu)\delta(\nu-\nu'),\end{aligned}$$

where we used (8). □

4. The Green's Function in the Free Space

Let us consider the Green's function $G(x,\mu;x_0,\mu_0)$ ($x,x_0\in(-\infty,\infty)$ and $\mu,\mu_0\in[-1,1]$) which obeys

$$\mu\frac{\partial}{\partial x}G(x,\mu;x_0,\mu_0)+G(x,\mu;x_0,\mu_0)=\frac{\varpi}{2}\int_{-1}^{1}G(x,\mu';x_0,\mu_0)\,d\mu'+\frac{1}{2\pi}\delta(x-x_0)\delta(\mu-\mu_0),\tag{15}$$

with

$$\lim_{|x|\to\infty}G(x,\mu;x_0,\mu_0)=0.\tag{16}$$

Equation (15) can be rewritten as

$$\mu\frac{\partial}{\partial x}G(x,\mu;x_0,\mu_0)+G(x,\mu;x_0,\mu_0)=\frac{\varpi}{2}\int_{-1}^{1}G(x,\mu';x_0,\mu_0)\,d\mu',$$

with the jump condition

$$G(x_0+0,\mu;x_0,\mu_0)-G(x_0-0,\mu;x_0,\mu_0)=\frac{1}{2\pi\mu}\delta(\mu-\mu_0).\tag{17}$$

Since singular eigenfunctions $\phi_\nu(\mu)$ form a complete set (Appendix D), we can expand $G(x,\mu;x_0,\mu_0)$ using $\phi_\nu(\mu)$. To satisfy (16) we look for a solution of the form

$$G(x,\mu;x_0,\mu_0)=\begin{cases}a_{0+}u_{0+}(x,\mu)+\displaystyle\int_0^1 A(\nu)u_\nu(x,\mu)\,d\nu, & x>x_0,\\ -a_{0-}u_{0-}(x,\mu)-\displaystyle\int_{-1}^0 A(\nu)u_\nu(x,\mu)\,d\nu, & x<x_0,\end{cases}$$

where $a_{0\pm}$ and $A(\nu)$ will be determined from (17). As $x\to x_0$,

$$G(x_0+0,\mu;x_0,\mu_0)=a_{0+}\phi_{0+}(\mu)e^{-x_0/\nu_0}+\int_0^1 A(\nu)\phi_\nu(\mu)e^{-x_0/\nu}\,d\nu,\quad x>x_0,$$

$$G(x_0-0,\mu;x_0,\mu_0)=-a_{0-}\phi_{0-}(\mu)e^{-x_0/\nu_0}-\int_{-1}^0 A(\nu)\phi_\nu(\mu)e^{-x_0/\nu}\,d\nu,\quad x<x_0.$$

Thus the jump condition reads

$$\frac{1}{2\pi\mu}\delta(\mu-\mu_0)=a_{0+}\phi_{0+}(\mu)e^{-x_0/\nu_0}+a_{0-}\phi_{0-}(\mu)e^{-x_0/\nu_0}+\int_{-1}^1 A(\nu)\phi_\nu(\mu)e^{-x_0/\nu}\,d\nu.\tag{18}$$

By multiplying (18) by $\mu\phi_\nu(\mu)$ and integrating over μ, we obtain

$$\begin{aligned}a_{0\pm}&=\frac{\pm1}{2\pi N_0}\phi_{0\pm}(\mu_0)e^{x_0/\nu_0},\\ A(\nu)&=\frac{1}{2\pi N(\nu)}\phi_\nu(\mu_0)e^{x_0/\nu_0}.\end{aligned}$$

The Green's function is obtained as

$$G(x,\mu;x_0,\mu_0) = \frac{1}{2\pi}\left[\frac{1}{N_0}\phi_{0\pm}(\mu_0)\phi_{0\pm}(\mu)e^{-|x-x_0|/\nu_0} \right.$$
$$\left. + \int_0^1 \frac{1}{N(\nu)}\phi_{\pm\nu}(\mu)\phi_{\pm\nu}(\mu_0)e^{-|x-x_0|/\nu}\,d\nu\right].$$

5. Diffusion Limit

Let us consider the specific intensity (angular flux) u for an isotropic source at the origin $\delta(x)/2\pi$. If $x \gg 1$ (diffusion limit), we have

$$u(x,\mu) = \int_{-1}^1 G(x,\mu;0,\mu_0)\,d\mu_0 \simeq \frac{\phi_0(\mu)}{2\pi N_0}e^{-x/\nu_0}.$$

The density $U(x)$ and current $J(x)$ are defined as

$$U(x) = 2\pi\int_{-1}^1 u(x,\mu)\,d\mu, \quad J(x) = 2\pi\int_{-1}^1 \mu u(x,\mu)\,d\mu.$$

When we observe a point x which is far away from the source, they are calculated as

$$U(x) \simeq \frac{1}{N_0}e^{-x/\nu_0} \equiv U_d(x),$$
$$J(x) \simeq \frac{1}{N_0}e^{-x/\nu_0}\int_{-1}^1 \mu\phi_0(\mu)\,d\mu = \frac{(1-\varpi)\nu_0}{N_0}e^{-x/\nu_0} \equiv J_d(x).$$

Note that $(\nu_0-\mu)\phi_0(\mu) = \varpi\nu_0/2$ implies $\int_{-1}^1 \mu\phi_0(\mu)\,d\mu = (1-\varpi)\nu_0$. Therefore we obtain

$$J_d(x) = -D\frac{dU_d(x)}{dx}, \quad D = (1-\varpi)\nu_0^2.$$

This is Fick's law of diffusion.

Appendix A. The Hölder Condition

Let $f(\mu)$ be a function on an arc L. The function $f(\mu)$ is said to satisfy a Hölder condition on L if for any two points μ, μ' of L

$$|f(\mu) - f(\mu')| \le C|\mu-\mu'|^\theta,$$

where $C > 0$ and $\theta \in (0,1]$ are constants.

Appendix B. Plemelj's Formulae

Let L be a smooth line or contour. Let $f(\nu)$ satisfy the Hölder condition on L. Suppose μ does not coincide with those end points at which $f(\mu) \neq 0$. Consider

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{f(\nu)}{\nu - z}\,d\nu.$$

Let $\Phi^\pm(\mu) = \Phi(\mu \pm 0)$ be the limiting values of $\Phi(z)$ from the left and from the right of L, respectively. If μ coincides with an end, where $f(\mu) = 0$, then $\Phi^+(\mu) = \Phi^-(\mu) = \Phi(\mu)$. We obtain Plemelj's formulae

$$\Phi^\pm(\mu) = \pm\frac{1}{2}f(\mu) + \frac{1}{2\pi i}\,\mathcal{P}\int_L \frac{f(\nu)}{\nu-\mu}\,d\nu, \tag{B.1}$$

or

$$\Phi^+(\mu) - \Phi^-(\mu) = f(\mu), \tag{B.2}$$

$$\Phi^+(\mu) + \Phi^-(\mu) = \frac{1}{i\pi}\,\mathcal{P}\!\int_L \frac{f(\nu)}{\nu-\mu}\,d\nu. \tag{B.3}$$

The proof is done with the residue theorem. We can formally express Plemelj's formulae as

$$\frac{1}{\nu-\mu\pm i0} = \frac{\mathcal{P}}{\nu-\mu} \mp i\pi\delta(\nu-\mu). \tag{B.4}$$

Appendix C. Poincaré-Bertrand Formula

Let L be a smooth arc or contour. Let $f(\nu,\nu')$ be a function of ν,ν' on this line L, satisfy the Hölder condition with respect to ν and ν'. We suppose μ is a fixed point on L not coinciding with one of its ends. We will show the Poincaré-Bertrand formula,

$$\begin{aligned}\mathcal{P}\!\int_L \frac{1}{\nu-\mu}\left(\mathcal{P}\!\int_L \frac{f(\nu,\nu')}{\nu'-\nu}\,d\nu'\right)d\nu &= -\pi^2 f(\mu,\mu)\\ &\quad + \mathcal{P}\!\int_L \left(\mathcal{P}\!\int_L \frac{f(\nu,\nu')}{(\nu-\mu)(\nu'-\nu)}\,d\nu\right)d\nu'.\end{aligned} \tag{C.1}$$

Equation (C.1) implies (14).

To prove (C.1), let us introduce

$$\Phi(z) = \int_L \frac{1}{\nu-z}\left(\mathcal{P}\!\int_L \frac{f(\nu,\nu')}{\nu'-\nu}\,d\nu'\right)d\nu,$$

$$\Psi(z) = \int_L \left(\mathcal{P}\!\int_L \frac{f(\nu,\nu')}{(\nu-z)(\nu'-\nu)}\,d\nu\right)d\nu',$$

where $z \notin L$. Since z is not on L, we can change the order of integrals, and we have

$$\Phi(z) = \Psi(z). \tag{C.2}$$

To consider (C.2), for the moment we assume that L is the segment $[0,\ell]$ on the real axis. Then $0 \le \nu \le \ell$ and $0 \le \nu' \le \ell$ cover the square region $Q = [0,\ell]\times[0,\ell]$. We consider the strip q whose area is calculated as

$$\int_0^\ell \left(\int_{\max(\nu-\varepsilon,0)}^{\min(\nu+\varepsilon,\ell)} d\nu'\right)d\nu = \int_0^\ell \left(\int_{\max(\nu'-\varepsilon,0)}^{\min(\nu'+\varepsilon,\ell)} d\nu\right)d\nu' = 2\varepsilon\ell - \varepsilon^2.$$

We can express $\Phi(z)$ and $\Psi(z)$ as

$$\Phi(z) = I_0 + I_1, \quad \Psi(z) = I_0 + I_2,$$

where

$$I_0 = \int_{Q-q} \frac{f(\nu,\nu')}{(\nu-z)(\nu'-\nu)}\,d\nu d\nu',$$

$$I_1 = \int_0^\ell \frac{1}{\nu-z}\left(\int_{\max(\nu-\varepsilon,0)}^{\min(\nu+\varepsilon,\ell)} \frac{f(\nu,\nu')}{\nu'-\nu}\,d\nu'\right)d\nu,$$

$$I_2 = \int_0^\ell \left(\int_{\max(\nu'-\varepsilon,0)}^{\min(\nu'+\varepsilon,\ell)} \frac{f(\nu,\nu')}{(\nu-z)(\nu'-\nu)}\,d\nu\right)d\nu'.$$

By noticing, for example,

$$\int_{\max(\nu-\varepsilon,0)}^{\min(\nu+\varepsilon,\ell)} \frac{f(\nu,\nu')}{\nu'-\nu}\, d\nu' = \int_{\max(\nu-\varepsilon,0)}^{\min(\nu+\varepsilon,\ell)} \frac{f(\nu,\nu')-f(\nu,\nu)}{\nu'-\nu}\, d\nu' + f(\nu,\nu)\int_{\max(\nu-\varepsilon,0)}^{\min(\nu+\varepsilon,\ell)} \frac{1}{\nu'-\nu}\, d\nu',$$

we see that $I_1 \to 0$ and $I_2 \to 0$ as $\varepsilon \to 0$. This discussion can be readily generalized to an arbitrary smooth line L. Thus we have (C.2).

By virtue of the Plemelj formula (B.3), we have

$$\Phi^+(\mu) + \Phi^-(\mu) = 2\,\mathcal{P}\!\!\int_L \frac{1}{\nu-\mu}\left(\mathcal{P}\!\!\int_L \frac{f(\nu,\nu')}{\nu'-\nu}\, d\nu'\right) d\nu. \tag{C.3}$$

Furthermore we write $\Psi(z)$ as

$$\Psi(z) = \int_L \frac{\psi(\nu',z)}{\nu'-z}\, d\nu',$$

where

$$\psi(\nu',z) = \int_L \left(\frac{1}{\nu-z} - \frac{1}{\nu-\nu'}\right) f(\nu,\nu')\, d\nu.$$

Define $\psi^\pm(\nu',\mu) = \psi(\nu',\mu\pm 0)$. By Plemelj's formulae (B.2) and (B.3), we obtain

$$\begin{aligned} \psi^+(\nu',\mu) - \psi^-(\nu',\mu) &= 2\pi i f(\mu,\nu'),\\ \psi^+(\nu',\mu) + \psi^-(\nu',\mu) &= 2\,\mathcal{P}\!\!\int_L \left(\frac{1}{\nu-\mu} - \frac{1}{\nu-\nu'}\right) f(\nu,\nu')\, d\nu\\ &= 2(\nu'-\mu)\,\mathcal{P}\!\!\int_L \frac{f(\nu,\nu')}{(\nu-\mu)(\nu'-\nu)}\, d\nu. \end{aligned}$$

We note that

$$\begin{cases} \psi(\nu',z) = \psi^+(\nu',\mu) + \varepsilon^+ & \text{if } z \text{ is to the left of } L,\\ \psi(\nu',z) = \psi^-(\nu',\mu) + \varepsilon^- & \text{if } z \text{ is to the right of } L, \end{cases}$$

where $\varepsilon^\pm \to 0$ as $z \to \mu$. We see that

$$\int_L \frac{\varepsilon^\pm}{\nu'-z}\, d\nu' \to 0,$$

when $z \to \mu$ along a straight line forming a finite angle with the tangent at μ. Using this fact together with Plemelj's formulae (B.1), we obtain

$$\Psi^\pm(\mu) = \pm i\pi\psi^\pm(\mu,\mu) + \mathcal{P}\!\!\int_L \frac{\psi^\pm(\nu',\mu)}{\nu'-\mu}\, d\nu'.$$

Hence,

$$\Psi^+(\mu) + \Psi^-(\mu) = -2\pi^2 f(\mu,\mu) + 2\,\mathcal{P}\!\!\int_L \left(\mathcal{P}\!\!\int_L \frac{f(\nu,\nu')}{(\nu-\mu)(\nu'-\nu)}\, d\nu\right) d\nu'. \tag{C.4}$$

By (C.2), (C.3), and (C.4), we obtain (C.1). □

Appendix D. Completeness

Definition. *Let $f(\mu)$ on $[-1,1]$ satisfy the Hölder condition. We further suppose that $f(\mu)$ is bounded everywhere on $[-1,1]$ with the possible exception of a finite number of points $t_1, t_2, \ldots$; however*

$$|f(\mu)| \leq \frac{C}{|\mu - t|^{\alpha}},$$

where $C > 0$, $0 < \alpha < 1$, and t stands for any of $t_1, t_2, \ldots$. Then we say $f(\mu)$ is in class G.

Theorem Appendix D.1. *(Completeness). The functions $\phi_{0\pm}(\mu)$ and $\phi_\nu(\mu)$ are complete for functions $f(\mu)$ of class G on $[-1,1]$.*

Proof. Let $f(\mu)$ be a given function of class G. Let $a_{0\pm}$ be constants and $A(\nu)$ be a function of class G. We will show that for any $f(\mu)$ there exist $a_{0\pm}$ and $A(\nu)$ such that

$$f(\mu) = a_{0+}\phi_{0+}(\mu) + a_{0-}\phi_{0-}(\mu) + \int_{-1}^{1} A(\nu)\phi_\nu(\mu)\, d\nu. \tag{D.1}$$

Let us first assume that we have a function $f(\mu)$ in the form

$$f(\mu) = \int_{-1}^{1} A(\nu)\phi_\nu(\mu)\, d\nu. \tag{D.2}$$

Using (6) and (10), we have

$$f(\mu) = \frac{1}{2}\left[\Lambda^+(\mu) + \Lambda^-(\mu)\right] A(\mu) + \frac{\varpi}{2}\, \mathcal{P}\!\int_{-1}^{1} \frac{\nu A(\nu)}{\nu - \mu}\, d\nu.$$

Let us define

$$n(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\varpi}{2}\frac{\nu A(\nu)}{\nu - z}\, d\nu \quad z \in \mathbb{C}. \tag{D.3}$$

We note that $n(z)$ is analytic in $\mathbb{C}$ excluding $[-1,1]$ and $n(z) = O\left(\frac{1}{|z|}\right)$ as $|z| \to \infty$. Also by the Plemelj formulae (B.1), we have for $\mu \in (-1,1)$,

$$n^+(\mu) = n(\mu + 0) = \frac{\varpi}{4}\mu A(\mu) + \frac{1}{2\pi i}\, \mathcal{P}\!\int_{-1}^{1} \frac{\varpi}{2}\frac{\nu A(\nu)}{\nu - \mu}\, d\nu,$$

$$n^-(\mu) = n(\mu - 0) = -\frac{\varpi}{4}\mu A(\mu) + \frac{1}{2\pi i}\, \mathcal{P}\!\int_{-1}^{1} \frac{\varpi}{2}\frac{\nu A(\nu)}{\nu - \mu}\, d\nu.$$

We readily have

$$n^+(\mu) + n^-(\mu) = \frac{1}{i\pi}\, \mathcal{P}\!\int_{-1}^{1} \frac{\varpi}{2}\frac{\nu A(\nu)}{\nu - \mu}\, d\nu,$$

$$n^+(\mu) - n^-(\mu) = \frac{\varpi}{2}\mu A(\mu).$$

Thus,

$$\frac{\varpi\mu}{2} f(\mu) = \frac{1}{2}\left[\Lambda^+(\mu) + \Lambda^-(\mu)\right]\left[n^+(\mu) - n^-(\mu)\right] + \frac{i\pi\varpi\mu}{2}\left[n^+(\mu) + n^-(\mu)\right].$$

Using (9), we have

$$\frac{\varpi\mu}{2} f(\mu) = \Lambda^+(\mu)n^+(\mu) - \Lambda^-(\mu)n^-(\mu). \tag{D.4}$$

We define

$$J(z)=\Lambda(z)n(z)-\frac{1}{2\pi i}\int_{-1}^{1}\frac{\varpi\mu}{2}\frac{f(\mu)}{\mu-z}\,d\mu.$$

Note that $J(z)$ is analytic in $\mathbb{C}$ excluding $[-1,1]$ and $J(z)=O\left(\frac{1}{|z|}\right)$ as $|z|\to\infty$. Using (D.4), we have

$$J^{+}(\mu)-J^{-}(\mu)=\Lambda^{+}(\mu)n^{+}(\mu)-\Lambda^{-}(\mu)n^{-}(\mu)-\frac{\varpi\mu}{2}f(\mu)=0.$$

That is, $J(z)$ has no discontinuity across the cut $(-1,1)$, and hence is analytic everywhere. Therefore, from Liouville's theorem,

$$J(z)=0\quad z\in\mathbb{C},$$

or

$$n(z)=\frac{1}{2\pi i\Lambda(z)}\int_{-1}^{1}\frac{\varpi\mu}{2}\frac{f(\mu)}{\mu-z}\,d\mu. \tag{D.5}$$

Here we notice a contradiction. Although $n(z)$ is analytic in $\mathbb{C}$ excluding $[-1,1]$, $n(z)$ in (D.5) has poles at $\pm\nu_0$ (see (7)) unless

$$\int_{-1}^{1}\frac{\varpi\mu}{2}\frac{f(\mu)}{\mu\pm\nu_0}\,d\mu=0. \tag{D.6}$$

Equation (D.6) does not hold in general. The first assumption (D.2) must be modified. We adopt (D.1). Then (D.6) can be written as

$$\int_{-1}^{1}\frac{\mu f(\mu)}{\mu\pm\nu_0}\,d\mu=\int_{-1}^{1}\frac{\mu}{\mu\pm\nu_0}\left[a_{0+}\phi_{0+}(\mu)+a_{0-}\phi_{0-}(\mu)\right]d\mu. \tag{D.7}$$

The right-hand side of (D.7) is expressed as

$$\begin{aligned}\int_{-1}^{1}\frac{\mu f(\mu)}{\mu+\nu_0}\,d\mu&=\frac{2}{\varpi\nu_0}\int_{-1}^{1}\mu\left[a_{0+}\phi_{0-}(\mu)\phi_{0+}(\mu)+a_{0-}\phi_{0-}(\mu)\phi_{0-}(\mu)\right]d\mu\\&=-\frac{2}{\varpi\nu_0}a_{0-}N_0,\\\int_{-1}^{1}\frac{\mu f(\mu)}{\mu-\nu_0}\,d\mu&=\frac{-2}{\varpi\nu_0}\int_{-1}^{1}\mu\left[a_{0+}\phi_{0+}(\mu)\phi_{0+}(\mu)+a_{0-}\phi_{0+}(\mu)\phi_{0-}(\mu)\right]d\mu\\&=-\frac{2}{\varpi\nu_0}a_{0+}N_0.\end{aligned}$$

Thus we obtain

$$a_{0\pm}=\frac{\pm1}{N_0}\int_{-1}^{1}\mu\phi_{0\pm}(\mu)f(\mu)\,d\mu.$$

With these $a_{0\pm}$, (D.1) holds for any $f(\mu)$. □