Quiz: Review of linear algebra

Solutions to this quiz must be submitted on **Mon, Sep 23** at the beginning of the class.

We consider the following system of linear equations which has n unknowns $x_1, \ldots, x_n$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{cases}$$

We can write the system as

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Problem 1 The ith row of $Ax = b$ is written as

$$\sum_{j=1}^{n} \underline{\qquad} = b_i,$$

where j is the column index. Fill the space on the left hand side.

Problem 2

(a) Suppose $n = 2$. What is the determinant of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$?

(b) Suppose $n = 3$. Write the determinant $\det A$.

(c) Find $\det B$:

$$B = \begin{pmatrix} 4 & 5 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 2 & 7 & 1 & 4 \\ 1 & 8 & 2 & 3 \end{pmatrix}.$$

In addition to the solution, explain how you calculate the determinant.

Problem 3

(a) Suppose $n = 2$ and A is invertible (i.e., A^{-1} exists). What is the inverse of A?

(b) For general n, how is A^{-1} given?

Problem 4 Which of the following matrices are invertible? Justify your answer. For those matrices that are not invertible, find a vector $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$.

$$\text{(a)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{(b)} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \text{(c)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \text{(d)} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \text{(e)} \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 0 & 3 & 3 \end{pmatrix}.$$

If A is invertible, we can obtain $\mathbf{x}$ by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

In Matlab, you can type `x=A\b`.

Problem 5 Find all real eigenvalues.

$$\text{(a)} \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 2 & 2 & 0 \end{pmatrix}.$$

For an invertible matrix A, the following conditions are equivalent.

(i) $A\mathbf{x} = \mathbf{b}$ has a unique solution for $\forall \mathbf{b}$.

(ii) A is invertible.

(iii) $\det A \neq 0$.

(iv) $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.

(v) The columns of A are linearly independent.

(vi) The eigenvalues of A are nonzero.

Problem 6 Prove one of the relations, (i) $\Longleftrightarrow$ (ii), (ii) $\Longleftrightarrow$ (iii), (iv) $\Longleftrightarrow$ (vi), etc.

Suppose A is invertible. Note that $\mathbf{x} = A^{-1}\mathbf{b}$ is not the best way to numerically compute $\mathbf{x}$. We will study two types of methods for solving $A\mathbf{x} = \mathbf{b}$: direct methods and iterative methods.