Problem Set 7 (11/1, 4, 6, 8, 11, 13, 15, 18)
Due on Wed, Nov 27

1) Consider $f(x) = \sin x$ for $-4\pi \leq x \leq 4\pi$. Find the Taylor series for $f(x)$ about $x = 0$ up to the x^7-term. The Taylor polynomial $p_n(x)$ is an approximation to $f(x)$.
   (a) Using Matlab, plot $f(x)$ and the Taylor polynomials $p_n(x)$ of degree $n = 1, 3, 5, 7$. Use `subplot`, and put $f(x)$ and a Taylor polynomial in each frame. Use the command `axis([-4*pi 4*pi -2 2])` to set the limits on the axes. Label each curve. Turn in the resulting plot.
   (b) For a given value of n, is the approximation valid for all values of x? Justify your answer by reference to the plots.
   (c) Does the approximation improve as the degree n increases? Justify your answer by reference to the plots.
2) Let $f(x) = e^{-|x|}$ and consider the three points $x_0 = -1, x_1 = 0, x_2 = 1$.
   (a) Find Newton's form for the interpolating polynomial $p_2(x)$.
   (b) Find the standard form for the interpolating polynomial $p_2(x)$.
   (c) Plot $f(x)$ and $p_2(x)$ on the same graph and label each curve. Sketch by hand or use Matlab.
   (d) Compute $\int_{-1}^{1} f(x)dx$ and $\int_{-1}^{1} p_2(x)dx$.
3) Let us consider the interpolating polynomial $p_n(x)$ of degree $\leq n$ that interpolates a given function $f(x)$ at a set of distinct points, $x_0, x_1, \ldots, x_n$, i.e., such that $p_n(x_i) = f(x_i)$ for $i = 0, \ldots, n$. The template below plots $f(x)$ and $p_n(x)$ for $n = 4, 8, 16$, for uniform points and Chebyshev points, on the interval $-1 \leq x \leq 1$. Your assignment is to fill in the template, run the code for the functions
   (a) $f(x) = e^{-|x|}$ and
   (b) $f(x) = e^{-x^2}$,

   turn in the output plots, and answer questions (b) and (c) from problem 1) above. Can you explain any difference in the results for the two examples of $f(x)$?
4) Plot $f(x) = \dfrac{1}{1+9x^2}$ and its natural cubic spline interpolant $s(x)$ for $n = 2, 4, 6, 8$, where $-1 \leq x \leq 1$, $x_i = -1 + ih$, $h = \dfrac{2}{n}$, $i = 0, \ldots, n$.

```
function interpolation
%
% This is a template for Problem 3).
%
clear; clf;
%
for k = 1:6
    if k==1; n = 4; itype=1; text='uniform points , n=4'; end
    if k==2; n = 4; itype=2; text='Chebyshev points , n=4'; end
    if k==3; n = 8; itype=1; text='uniform points , n=8'; end
    if k==4, n = 8; itype=2; text='Chebyshev points , n=8'; end
    if k==5, n = 16; itype=1; text='uniform points , n=16'; end
    if k==6, n = 16; itype=2; text='Chebyshev points , n=16'; end
%
    if itype==1; h = ...; x = ...; end
    if itype==2; h = ...; x = ...; end
%
    f = ...;
    coeff = polyfit(x,f,n); % polyfit computes the coefficients of p_n(x)
%
% Plot f(x) and p_n(x) on a fine mesh.
%
    h = 0.001; x = -1.5:h:1.5;
    f = ...;
    p = polyval(coeff,x); % polyval evaluates p_n(x)
%
    subplot(3,2,k)
    plot(x,f,x,p,'--'); axis([-1.5 1.5 -1.5 1.5])
    title(text)
end
```