Problem Set 4 (9/30, 10/2, 4, 7, 9, 11)
Due on Fri, Oct 18

1) Consider the equations

$$2x_1 + 3x_2 - x_3 = 5, \quad 4x_1 + 4x_2 - 3x_3 = 3, \quad -2x_1 + 3x_2 - x_3 = 1.$$

(a) Write the system in the form $(\,A\,|\mathbf{b})$ and find the LU factorization of A.
(b) Solve for $\mathbf{x}$ by forward and back substitution. That is, first obtain $\mathbf{y}$, where $\mathbf{y} = U\mathbf{x}$, and then obtain $\mathbf{x}$.

2) Let A be a 3×3 matrix. Suppose we apply LU factorization with partial pivoting and obtain $E_2P_2E_1P_1A = U$, where U is an upper triangular matrix and

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix}, \; E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{pmatrix}, \; P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Compute $\tilde{E}_1 = P_2E_1P_2$.
(b) Show that $P_2E_1 = \tilde{E}_1P_2$. Note that this implies $E_2\tilde{E}_1P_2P_1A = U$.
(c) Compute $P = P_2P_1$ and $L = \tilde{E}_1^{-1}E_2^{-1}$.
(d) Show that $PA = LU$.
(e) Find P, L, U such that $PA = LU$ and use the factorization to solve $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

3) Consider the linear system

$$2x_1 + x_2 = 1, \quad x_1 + 2x_2 = -1.$$

The exact solution is $x_1 = 1$, $x_2 = -1$.

(a) Write the system in matrix form and solve it by LU factorization.
(b) Write Jacobi's method in component form and take three steps ($k = 1, 2, 3$) starting from initial guess $x_1^{(0)} = x_2^{(0)} = 0$. Complete the table below.

k	$x_1^{(k)}$	$x_2^{(k)}$	$\|\mathbf{e}_k\|_\infty$	$\|\mathbf{e}_k\|_\infty / \|\mathbf{e}_{k-1}\|_\infty$
1	*	*	*	*
⋮	⋮	⋮	⋮	⋮

(c) Repeat (b) for the Gauss-Seidel method.

4) Let us solve $-y'' = 25\cos(\pi x)$, $y(0) = 0$, $y(1) = 1$ using the LU factorization. Take the mesh points as $x_i = ih$, $h = 1/(n+1)$, $i = 0, 1, \ldots, n+1$.

(a) The exact solution has the form $y(x) = \frac{25}{\pi^2}\cos(\pi x) + ax + b$. Find a and b.

(b) Find a tridiagonal linear system $A\mathbf{w} = \mathbf{r}$.

(c) Write a Matlab code. Do not create the full matrix A in the code, but instead use vectors to store the nonzero matrix elements and numerical solution $\mathbf{w}$. This saves memory and is important in realistic applications. You can use the template code below.

(d) Run the code. Make figures for four values of the mesh size, $h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$. Plot the exact solution and numerical solution in each figure.

(e) Also produce a table displaying the errors $\|\mathbf{y}-\mathbf{w}\|_\infty$, $\|\mathbf{y}-\mathbf{w}\|_\infty/h$, $\|\mathbf{y}-\mathbf{w}\|_\infty/h^2$, and $\|\mathbf{y}-\mathbf{w}\|_\infty/h^3$.

```
%
% Template for Problem 4)
% -y''=r, y(0)=alpha, y(1)=beta
%
function bvp1d
clear; clf;
alpha = 0; beta = 1;                    % boundary conditions
for icase=1:4
    n = 2^icase-1; h = 1/(n+1);         % h = mesh size
    xe = 0:0.0025:1;                    % fine mesh for plotting exact solution
    ye = ...;                           % exact solution on fine mesh
% Set up for numerical solution.
for i=1:n
    xh(i) = i*h;                        % mesh points
    yh(i) = ...;                        % exact solution at mesh points
    a(i) = ...; b(i) = ...; c(i) = ...; % matrix elements
    r(i) = ...;                         % right hand side vector
end
    r(1) = ...;                         % adjust for BC at x=0
    r(n) = ...;                         % adjust for BC at x=1
    wh = LU_fb(a,b,c,r);                % numerical solution
% output
    table(icase,1) = h;
    table(icase,2) = norm(yh-wh,inf);
    table(icase,3) = norm(yh-wh,inf)/h;
    table(icase,4) = norm(yh-wh,inf)/h^2;
    table(icase,5) = norm(yh-wh,inf)/h^3;
    xplot = [0 xh 1]; wplot = [alpha wh beta];
    subplot(2,2,icase); plot(xe,ye,xplot,wplot,'-o');
    string = sprintf('h=1/%d',n+1); title(string)
end
table
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function w = LU_fb(a,b,c,r)
% input:  a, b, c, r - matrix elements and right hand side vector
% output:  w - solution of linear system
n = length(r);
%
% Fill in the steps below using the tridiagonal LU method given in class.
%
% find L, U
%
% solve Lz = r
%
% solve Uw = z
%
```