Problem Set 3 (9/20, 23, 25, 27)
Due on Fri, Oct 4

1) Consider the system

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 5, \\ 4x_1 + 4x_2 - 3x_3 = 3, \\ -2x_1 + 3x_2 - x_3 = 1. \end{cases}$$

(a) Write the augmented matrix $(\ A\ |b)$ and find x_1, x_2, x_3 by Gaussian elimination (no pivoting). Show intermediate steps.

(b) Compute the determinant of A in two ways: the usual method and the formula $\det A = a_{11}^{(1)} a_{22}^{(2)} a_{33}^{(3)}$, where $a_{kk}^{(k)}$ is the pivot element in step k of Gaussian elimination.

2) Consider $A = \begin{pmatrix} -2 & 1 \\ 2 & 0 \end{pmatrix}$.

(a) Find $\dfrac{\|A\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty}$ for the following three vectors.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b) Find a vector $\mathbf{x}$ such that $\dfrac{\|A\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} = \|A\|_\infty$.

3) Let $A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix}$.

(a) Is A invertible? Explain.

(b) Show that $\mathbf{x}$ is the exact solution to $A\mathbf{x} = \mathbf{b}$.

(c) Let $\tilde{\mathbf{x}}_1 = \begin{pmatrix} 2.1 \\ -2.1 \end{pmatrix}$, $\tilde{\mathbf{x}}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\tilde{\mathbf{x}}_3 = \begin{pmatrix} 0.9911 \\ -0.4870 \end{pmatrix}$. The vectors $\tilde{\mathbf{x}}_1$, $\tilde{\mathbf{x}}_2$,$\tilde{\mathbf{x}}_3$ are approximations to the exact solution $\mathbf{x}$. For each case, find the error norm $\|\mathbf{e}\|_\infty = \|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty$ and residual norm $\|\mathbf{r}\|_\infty = \|\mathbf{b} - A\tilde{\mathbf{x}}\|_\infty$.

(d) In part (c), which case has the smallest error norm? Which case has the smallest residual norm? Does a smaller error norm imply a smaller residual norm? Does a smaller residual norm imply a smaller error norm?

(e) Find $\|A\|_\infty$, $\|A^{-1}\|_\infty$, and $\kappa_\infty(A)$.

(f) For the vectors $\tilde{\mathbf{x}}_1$, $\tilde{\mathbf{x}}_2$, $\tilde{\mathbf{x}}_3$, show that the following relation is satisfied.

$$\frac{\|\mathbf{e}\|_\infty}{\|\mathbf{x}\|_\infty} \le \kappa_\infty(A) \frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}.$$

(g) Derive the above relation $\frac{\|\mathbf{e}\|_\infty}{\|\mathbf{x}\|_\infty} \le \kappa_\infty(A) \frac{\|\mathbf{r}\|_\infty}{\|\mathbf{b}\|_\infty}$.

4) Solve the system below in three ways.

$$\begin{cases} 3.41x_1 + 1.23x_2 - 1.09x_3 = 4.72, \\ 2.71x_1 + 2.14x_2 + 1.29x_3 = 3.10, \\ 1.89x_1 - 1.91x_2 - 1.89x_3 = 2.91. \end{cases}$$

(a) Gaussian elimination with no pivoting, 3 decimal digit arithmetic with rounding (not chopping).

(b) Gaussian elimination with partial pivoting, 3 decimal digit arithmetic with rounding (not chopping).

(c) Matlab backslash command.

5) Verify that the l_∞-norm $\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$ satisfies the properties of a vector norm:

(a) $\|\mathbf{x}\| \ge 0$, and $\|\mathbf{x}\| = 0 \;\Leftrightarrow\; \mathbf{x} = \mathbf{0}$.

(b) $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|, \quad \alpha \in \mathbb{C}$.

(c) $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.

6) For matrices A and B, prove $\|AB\| \le \|A\|\,\|B\|$ (*Hint:* Consider $\|AB\mathbf{x}\|$).