Problem Set 1 (9/4, 6, 9)
Due on Fri, Sep 13

1) (a) Convert $(9.125)_{10}$ to base 2.
   (b) Convert $(110110.001)_2$ to base 10.

2) Consider the floating point representation $x = \pm(0.d_1 d_2 \dots d_n)_\beta \cdot \beta^e$, where $d_1 \neq 0$, $0 \le d_i \le \beta - 1$, $-M \le e \le M$. Suppose $\beta = 2$, $n = 5$, and $M = 2$.
   (a) What is the largest number $x_{\max}$?
   (b) What is the smallest positive number $x_{\min}$?
   (c) How many different numbers can be represented?
   (d) Find $\mathrm{fl}(\sqrt{2})$, the floating point representation of $\sqrt{2}$ in this system. Then convert the result to decimal form.

3) Matlab gives `pi` $= 3.141592653589793$ and $355/113 = 3.141592920353983$. All the digits shown are correct. (Use the command `format long` to see all the digits). Matlab also gives `pi` $-$ $(355/113) = -2.667641894049666\text{e}{-07}$; do you trust all the digits in this result? Explain your answer.

4) Consider the equation $x^2 + 25x + 0.1 = 0$. In general, for $ax^2 + bx + c = 0$, x is obtained by using the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
   (a) Solve for the roots using the quadratic formula. Find x using Matlab.
   (b) Suppose you have a 4-digit computer with the base 10 (i.e., each arithmetic step is rounded to 4 digits). Find x by the quadratic formula.
   (c) Show that $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{2c}{-b \mp \sqrt{b^2 - 4ac}}$.
   (d) Find x with 4-digit arithmetic by using the new formula. Do the results change? Explain.

5) Let $f(x) = \sqrt{1 + x^2} - 1$. Matlab gives $f(0.1) = 0.004987562112089$.
   (a) Evaluate $f(x)$ for $x = 0.1$ using 4-digit arithmetic. Show all intermediate steps.
   (b) Show that $f(x) = x^2/(\sqrt{1 + x^2} + 1)$.
   (c) Evaluate $x^2/(\sqrt{1 + x^2} + 1)$ for $x = 0.1$ using 4-digit arithmetic. Show all intermediate steps. Is the result improved?

6) The forward difference approximation for $f'(x)$ is $D_+ f(x) = \dfrac{f(x+h) - f(x)}{h}$. Similarly the centered difference approximation for $f'(x)$ is $D_0 f(x) = \dfrac{f(x+h) - f(x-h)}{2h}$. We consider $f(\pi/4)$, where $f(x) = \sin x$. Note that $f'(\pi/4) = \cos(\pi/4) = 0.70710678$.
   (a) Present a table in the format below; take $h = 0.1, 0.05, 0.025, 0.0125$; the first line for $h = 0.1$ is given and you must fill in the entries for the remaining values of h.

h	$D_+f(x)$	$\lvert f'(x) - D_+f(x)\rvert$	$\dfrac{\lvert f'(x) - D_+f(x)\rvert}{h}$	$\dfrac{\lvert f'(x) - D_+f(x)\rvert}{h^2}$	$\dfrac{\lvert f'(x) - D_+f(x)\rvert}{h^3}$
0.1	0.67060297	0.03650381	0.3650381	3.650381	36.50381
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(b) Present a table for $D_0f(x)$ in the same format as part (a). Which approximation is more accurate, $D_+f(x)$ or $D_0f(x)$?

(c) Using Taylor series, show that $D_0f(x) = f'(x) + ch^2 + \cdots$ for some constant c which is independent of h. Recall that $D_+f(x)$ is first order accurate (i.e., the truncation error is $O(h)$). What is the order of accuracy of $D_0f(x)$?

(d) Modify the Matlab code given in class to plot the error in $D_+f(x)$ and $D_0f(x)$ for step size $h = 1/2^{(j-1)}$ with $j = 1 : 65$. Use log scales for the error $|f'(x) - Df(x)|$ and the step size h. Plot both cases on the same graph (to do this in Matlab, type `hold on` after the first `loglog` command).

7) The backward finite-difference operator is defined by $D_-f(x) = \dfrac{f(x) - f(x-h)}{h}$.

(a) Show that $D_+D_-f(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$.

(b) Using Taylor series, show that $D_+D_-f(x) = f''(x) + ch^2 + \cdots$, and find the constant c.