MATH 454 SECTION 002

Quiz 9

April 4, 2014, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Let $f(s) = 0$ for $-1 < s < 0$ and $f(s) = 1$ for $0 < s < 1$. Find the expansion of $f(s)$ in a series of Legendre polynomials. [*Hint:* The Legendre polynomial $P_k(s)$ satisfies the Legendre equation, $\frac{d}{ds}\left[(1-s^2)\frac{d}{ds}P_k(s)\right] + k(k+1)P_k(s) = 0$. These Legendre polynomials satisfy the orthogonality relations, $\int_{-1}^{1} P_k(s)P_j(s)ds = \frac{2}{2k+1}\delta_{kj}$.]

Solution [6] We write [1]

$$f(s) = \sum_{k=0}^{\infty} A_k P_k(s), \qquad -1 < s < 1,$$

where A_k are constants to be determined. We multiply $P_j(s)$ and integrate over s [1]:

$$\int_{-1}^{1} f(s)P_j(s)ds = \sum_{k=0}^{\infty} A_k \int_{-1}^{1} P_k(s)P_j(s)ds.$$

The left-hand side is obtained as follows. For $j = 0$, we have [1]

$$\text{LHS} = \int_0^1 P_0(s)ds = \int_0^1 ds = 1.$$

For $j \geq 1$, we have [1]

$$\begin{aligned}\text{LHS} &= \int_0^1 P_j(s)ds = \int_0^1 \frac{-1}{j(j+1)}\frac{d}{ds}\left[(1-s^2)\frac{d}{ds}P_j(s)\right]ds \\ &= \frac{-1}{j(j+1)}\left.(1-s^2)\frac{d}{ds}P_j(s)\right|_0^1 = \frac{1}{j(j+1)}\left.\frac{dP_j(s)}{ds}\right|_{s=0}.\end{aligned}$$

The right-hand side is obtained as [1]

$$\text{RHS} = \sum_{k=0}^{\infty} A_k \int_{-1}^{1} P_k(s)P_j(s)ds = \sum_{k=0}^{\infty} A_k \frac{2}{2k+1}\delta_{kj} = \frac{2A_j}{2j+1}.$$

Therefore we obtain

$$A_0 = \frac{1}{2}, \qquad A_j = \frac{2j+1}{2j(j+1)}P_j'(0) \quad (j \geq 1).$$

That is, [1]

$$f(s) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2k+1}{2k(k+1)}P_k'(0)P_k(s), \qquad -1 < s < 1.$$