MATH 454 SECTION 002

Quiz 8

April 2, 2014, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Find the solution of the heat equation $u_t = K\nabla^2 u$ in the infinite cylinder $0 \le \rho < \rho_{\max}$ satisfying the boundary condition $u(\rho_{\max}, \varphi, t) = 0$ and the initial condition $u(\rho, \varphi, 0) = \rho_{\max}^2 - \rho^2$. [*Hint:* The Laplacian in polar coordinates is given by $\Delta = \partial_{\rho\rho} + (1/\rho)\partial_\rho + (1/\rho^2)\partial_{\varphi\varphi}$. You can use the facts that $\Phi''(\varphi) + \mu\Phi(\varphi) = 0,\ \Phi(-\pi) = \Phi(\pi),\ \Phi'(-\pi) = \Phi'(\pi)$ is solved as $\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi,\ \mu = m^2,\ m = 0, 1, 2, \cdots$. We note that a_{mn}, b_{mn} satisfy $1 - x^2 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(x x_n^{(m)})\left(a_{mn}\cos m\varphi + b_{mn}\sin m\varphi\right),\ 0 \le x < 1$, when $a_{0n} = 8/\left[x_n^{(0)3} J_1(x_n^{(0)})\right]$ $(J_0(x_n^{(0)}) = 0)$, $a_{mn} = b_{mn} = 0$ $(m \ge 1)$.]

Solution [8] We will solve

$$\begin{cases} u_t = K\left(u_{\rho\rho} + \dfrac{1}{\rho}u_\rho + \dfrac{1}{\rho^2}u_{\varphi\varphi}\right), & 0 \le \rho < \rho_{\max}, \quad t > 0, \\ u(\rho, \varphi, t) = 0, & \rho = \rho_{\max}, \quad t > 0, \\ u(\rho, \varphi, 0) = \rho_{\max}^2 - \rho^2, & 0 \le \rho < \rho_{\max}. \end{cases}$$

We look for separated solutions of the form [2] $u(\rho, \varphi, t) = R(\rho)\Phi(\varphi)T(t)$ (the form $u = RT$ is also fine because we can assume that u is independent of φ from the initial and boundary conditions). By introducing separation constants as $-\lambda = T'/(KT)$ and $-\mu = \Phi''/\Phi$, we obtain

$$\begin{aligned} &\Phi''(\varphi) + \mu\Phi(\varphi) = 0, \quad \Phi(-\pi) = \Phi(\pi), \quad \Phi'(-\pi) = \Phi'(\pi), \\ &R''(\rho) + \frac{1}{\rho}R'(\rho) + \left(\lambda - \frac{\mu}{\rho^2}\right)R(\rho) = 0, \quad R(\rho_{\max}) = 0, \\ &T'(t) + \lambda K T(t) = 0. \end{aligned}$$

The equation for Φ is solved as $\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi,\ \mu = m^2,\ m = 0, 1, 2, \cdots$. We obtain $T(t) = e^{-\lambda K t}$. From the second equation we obtain [2] $R(\rho) = J_m(\rho\sqrt{\lambda})$. Since $R(\rho_{\max}) = 0$, we obtain [2] $\sqrt{\lambda} = x_n^{(m)}/\rho_{\max}$ where $J_m(x_n^{(m)}) = 0$, $x_n^{(m)} > 0$. Hence the general solution is obtained as

$$u(\rho, \varphi, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\rho x_n^{(m)}}{\rho_{\max}}\right)\left(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi\right) e^{-(x_n^{(m)}/\rho_{\max})^2 K t}.$$

To satisfy the condition, A_{mn}, B_{mn} must satisfy

$$1 - x^2 = \frac{1}{\rho_{\max}^2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(x x_n^{(m)})\left(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi\right),$$

where $x = \frac{\rho}{\rho_{\max}}$. Since $A_{0n} = 8\rho_{\max}^2/\left[x_n^{(0)3} J_1(x_n^{(0)})\right]$, $A_{mn} = B_{mn} = 0$ $(m \ge 1)$, we obtain [2]

$$u(\rho, \varphi, t) = 8\rho_{\max}^2 \sum_{n=1}^{\infty} \frac{1}{x_n^{(0)3} J_1(x_n^{(0)})} J_0\left(\frac{\rho x_n^{(0)}}{\rho_{\max}}\right) e^{-(x_n^{(0)}/\rho_{\max})^2 K t}.$$