Quiz 8 – Supplement

Let us consider

$$1 - x^2 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(xx_n^{(m)})\left(a_{mn}\cos m\varphi + b_{mn}\sin m\varphi\right).$$

We can find a_{mn}, b_{mn} as follow. First of all, we look at the right-hand side as the Fourier series of $1-x^2$. We find

$$\sum_{n=1}^{\infty} J_0(xx_n^{(0)})a_{0n} = 1 - x^2,$$

and all other a_{mn}, b_{mn} are zero. For simplicity, hereafter, we write $x_n = x_n^{(0)}$. To find a_{0n}, we multiply another Bessel function and integrate over x:

$$\int_0^1 (1-x^2)J_0(xx_n)xdx = \int_0^1 \sum_{m=1}^{\infty} a_{0m}J_0(xx_m)J_0(xx_n)xdx.$$

We obtain

$$\begin{aligned}
\text{LHS} &= \frac{1}{x_n^4}\int_0^{x_n}(x_n^2 - t^2)J_0(t)tdt \qquad [t = xx_n] \\
&= \frac{1}{x_n^4}\left[x_n^2\, tJ_1(t)\big|_0^{x_n} - t^3J_1(t)\big|_0^{x_n} + 2\int_0^{x_n} t^2J_1(t)dt\right] \qquad [(tJ_1(t))' = tJ_0(t)] \\
&= \frac{2}{x_n^4}\left[-\, t^2J_0(t)\big|_0^{x_n} + 2\int_0^{x_n} tJ_0(t)dt\right] \qquad [J_0'(t) = -J_1(t)] \\
&= \frac{4}{x_n^4}\, tJ_1(t)\big|_0^{x_n} \\
&= \frac{4}{x_n^3}J_1(x_n).
\end{aligned}$$

Alternatively, using $(t^2J_2(t))' = t^2J_1(t)$,

$$\text{LHS} = \frac{2}{x_n^4}\int_0^{x_n} t^2J_1(t)dt = \frac{2}{x_n^4}\, t^2J_2(t)\big|_0^{x_n} = \frac{2}{x_n^2}J_2(x_n)$$

is also fine (cf. the recurrence relation $J_1(x) = \frac{x}{2}[J_0(x) + J_2(x)]$). On the other hand,

$$\begin{aligned}
\text{RHS} &= \sum_{m=1}^{\infty} a_{0m}\int_0^1 J_0(xx_m)J_0(xx_n)xdx \\
&= a_{0n}\int_0^1 J_0(xx_n)^2xdx \\
&= \frac{a_{0n}}{2}J_1(x_n)^2.
\end{aligned}$$

Therefore we obtain,

$$a_{0n} = \frac{8}{x_n^3J_1(x_n)}, \qquad J_0(x_n) = 0, \quad x_n > 0.$$