MATH 454 SECTION 002

Quiz 8

March 28, 2014, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Find the solution of the vibrating (circular) membrane problem (i.e., the edges are fixed) in the case where $u(\rho,\varphi,0) = 0$ and $u_t(\rho,\varphi,0) = 1$, $0 < \rho < a$. [*Hint:* The wave equation is written as $u_{tt} = c^2\Delta u$, where the Laplacian in polar coordinates is given by $\Delta = \partial_{\rho\rho}+(1/\rho)\partial_\rho+(1/\rho^2)\partial_{\varphi\varphi}$. You can use the facts that $\Phi''(\varphi)+\mu\Phi(\varphi) = 0$, $\Phi(-\pi) = \Phi(\pi)$, $\Phi'(-\pi) = \Phi'(\pi)$ is solved as $\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi$, $\mu = m^2$, $m = 0, 1, 2, \cdots$. Also note that $T''(t) + \lambda c^2 T(t) = 0$, $T(0) = 0$ is solved as $T(t) = \sin\left(ct\sqrt{\lambda}\right)$. Finally we note that A_{mn}, B_{mn} satisfy $1 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{cx_n^{(m)}}{a} J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)$ when $A_{0n} = \frac{2a}{c}\left[x_n^{(0)2}J_1(x_n^{(0)})\right]^{-1}$ $\left(J_0\left(x_n^{(0)}\right) = 0\right)$, $A_{mn} = B_{mn} = 0$ $(m \geq 1)$.]

Solution [8]

$$\begin{cases} u_{tt} = c^2\nabla^2 u = c^2\left(u_{\rho\rho} + \dfrac{1}{\rho}u_\rho + \dfrac{1}{\rho^2}u_{\varphi\varphi}\right), & 0 \leq \rho < a, \quad t > 0, \\ u(\rho,\varphi,t) = 0, & \rho = a, \quad t > 0, \\ u(\rho,\varphi,0) = 0, \quad u_t(\rho,\varphi,0) = 1, & 0 \leq \rho < a. \end{cases}$$

We look for separated solutions of the form [2] $u(\rho,\varphi,t) = R(\rho)\Phi(\varphi)T(t)$ (the form $u = RT$ is also fine because we can assume that u is independent of φ from the initial and boundary conditions). By $-\lambda = (1/c^2)T''/T$ and $-\mu = \Phi''/\Phi$, we obtain

$$\begin{aligned} &\Phi''(\varphi) + \mu\Phi(\varphi) = 0, \quad \Phi(-\pi) = \Phi(\pi), \quad \Phi'(-\pi) = \Phi'(\pi), \\ &R''(\rho) + \frac{1}{\rho}R'(\rho) + \left(\lambda - \frac{\mu}{\rho^2}\right)R(\rho) = 0, \quad R(a) = 0, \\ &T''(t) + \lambda c^2 T(t) = 0, \quad T(0) = 0. \end{aligned}$$

The equation for Φ is solved as $\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi$, $\mu = m^2$, $m = 0, 1, 2, \cdots$. The equation for T is solved as $T(t) = \sin\left(ct\sqrt{\lambda}\right)$. From the second equation we obtain [2] $R(\rho) = J_m(\rho\sqrt{\lambda})$. Since $R(a) = 0$, we obtain [2] $\sqrt{\lambda} = x_n^{(m)}/a$ where $J_m(x_n^{(m)}) = 0$, $x_n^{(m)} > 0$. The general solution is obtained as

$$u(\rho,\varphi,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)\sin\frac{ctx_n^{(m)}}{a}.$$

To satisfy the condition $u_t(\rho,\varphi,0) = 1$, A_{mn}, B_{mn} must satisfy

$$1 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{cx_n^{(m)}}{a}J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi).$$

Since $A_{0n} = (2a/c)\left[x_n^{(0)2}J_1(x_n^{(0)})\right]^{-1}$ $\left(J_0\left(x_n^{(0)}\right) = 0\right)$, $A_{mn} = B_{mn} = 0$ $(m \geq 1)$, we obtain [2]

$$u(\rho,\varphi,t) = \frac{2a}{c}\sum_{n=1}^{\infty}\frac{1}{x_n^2 J_1(x_n)}J_0\left(\frac{\rho x_n}{a}\right)\sin\frac{ctx_n}{a}.$$