MATH 454 SECTION 002

Quiz 7

March 14, 2014, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Solve the initial-value problem for the heat equation $u_t = K\nabla^2 u$ in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u(0,y,t) = 0$, $u_x(L_1,y,t) = 0$, $u(x,0,t) = 0$, $u_y(x,L_2,t) = 0$ and the initial condition $u(x,y,0) = 1$. [*Hint:* You can start with the general solution below without deriving it. If we write $u(x,y,t) = \phi_1(x)\phi_2(y)T(t)$, we can introduce separation constants as $\frac{T'}{T} = -\lambda K$, $\frac{\phi_1''}{\phi_1} = -\mu_1$, $\frac{\phi_2''}{\phi_2} = -\mu_2$, where $\lambda = \mu_1 + \mu_2$. You can use the facts that $\phi'' + \mu\phi = 0$, $\phi(0) = \phi'(L) = 0$ is solved as $\phi(x) = \phi^{(m)}(x) = \sin((m-\frac{1}{2})\pi x/L)$, $\mu = \mu^{(m)} = \left((m-\frac{1}{2})\pi/L\right)^2$ $(m = 1,2,\ldots)$. They satisfy $\int_0^L \phi^{(m)}(x)dx = L/[(m-\frac{1}{2})\pi]$ and $\int_0^L \left[\phi^{(m)}(x)\right]^2 dx = \frac{L}{2}$. The general solution is written as $u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\phi_1^{(m)}(x)\phi_2^{(n)}(y)e^{-\lambda_{mn}Kt}$, where $\lambda_{mn} = \left((m-\frac{1}{2})\pi/L_1\right)^2 + \left((n-\frac{1}{2})\pi/L_2\right)^2$.]

Solution [8] Let us write $u(x,y,t) = \phi_1(x)\phi_2(y)T(t)$. We obtain $\phi_1 = \phi_1^{(m)} = \sin\frac{(m-\frac{1}{2})\pi x}{L_1}$ $(m = 1,2,\ldots)$, $\phi_2 = \phi_2^{(n)} = \sin\frac{(n-\frac{1}{2})\pi y}{L_2}$ $(n = 1,2,\ldots)$, and $T(t) = e^{-\lambda K t}$. The general solution is written as

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\phi_1^{(m)}(x)\phi_2^{(n)}(y)e^{-\lambda_{mn}Kt},$$

where $\lambda_{mn} = \left((m-\frac{1}{2})\pi/L_1\right)^2 + \left((n-\frac{1}{2})\pi/L_2\right)^2$. By the initial condition we have [2]

$$1 = \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'}\phi_1^{(m')}(x)\phi_2^{(n')}(y).$$

We multiply $\phi_1^{(m)}(x)\phi_2^{(n)}(y)$ on both sides and integrate both sides over x, y [2]:

$$\int_0^{L_2}\int_0^{L_1} \phi_1^{(m)}(x)\phi_2^{(n)}(y)dxdy = \int_0^{L_2}\int_0^{L_1} \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'}\phi_1^{(m')}(x)\phi_2^{(n')}(y)\phi_1^{(m)}(x)\phi_2^{(n)}(y)dxdy.$$

$$\begin{aligned}
\text{LHS} &= \int_0^{L_2} \phi_2^{(n)}(y)dy \int_0^{L_1} \phi_1^{(m)}(x)dx = \frac{L_1}{(m-\frac{1}{2})\pi}\frac{L_2}{(n-\frac{1}{2})\pi}, \\
\text{RHS} &= \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'} \int_0^{L_1} \phi_1^{(m')}(x)\phi_1^{(m)}(x)dx \int_0^{L_2} \phi_2^{(n')}(y)\phi_2^{(n)}(y)dy = B_{mn}\frac{L_1}{2}\frac{L_2}{2},
\end{aligned}$$

where we used $\int_0^{L_1} \phi_1^{(m')}(x)\phi_1^{(m)}(x)dx = 0$ $(m' \neq m)$ and $\int_0^{L_2} \phi_2^{(n')}(y)\phi_2^{(n)}(y)dy = 0$ $(n' \neq n)$ from the Sturm-Liouville theory [2]. Finally we obtain [2]

$$u(x,y,t) = \frac{4}{\pi^2}\sum_{m,n=1}^{\infty} \frac{\sin[(m-\frac{1}{2})(\pi x/L_1)]}{m-\frac{1}{2}} \frac{\sin[(n-\frac{1}{2})(\pi y/L_2)]}{n-\frac{1}{2}} e^{-\lambda_{mn}Kt}.$$