# MATH 454 SECTION 002

## Quiz 6

February 28, 2014, Instructor: Manabu Machida

Name: ______________________________________________

---

- To receive full credit you must show all your work.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

---

Let us consider heat flow in a circular ring of circumference $L$. Solve the heat equation $u_t = Ku_{zz}$ ($K > 0$) satisfying the periodic boundary conditions $u(0,t) = u(L,t)$, $u_z(0,t) = u_z(L,t)$, and the initial conditions $u(z,0) = 100$ if $0 < z < L/2$ and $u(z,0) = 0$ if $L/2 < z < L$. [*Hint:* You can use the facts that $\phi'' + \lambda\phi = 0$, $\phi(0) = \phi(L)$, $\phi'(0) = \phi'(L)$ is solved as $\phi(z) = \phi_n(z) = A_n \cos(\sqrt{\lambda_n} z) + B_n \sin(\sqrt{\lambda_n} z)$, where $\lambda_n = (2n\pi/L)^2$ ($n = 0, 1, 2, \ldots$). Here $\int_0^L \phi_n(z)\phi_{n'}(z)dz = LA_0^2$ for $n = n' = 0$, $(L/2)(A_n^2 + B_n^2)\delta_{nn'}$ otherwise. For $n, m = 1, 2, \ldots$, you can also use the orthogonality relations $\int_0^L \sin(2n\pi x/L) = \int_0^L \cos(2n\pi x/L) = 0$, $\int_0^L \sin(n\pi x/L)\sin(m\pi x/L)dx = (L/2)\delta_{nm}$, $\int_0^L \cos(n\pi x/L)\cos(m\pi x/L)dx = (L/2)\delta_{nm}$, $\int_0^L \sin(n\pi x/L)\cos(m\pi x/L)dx = 2Ln/[\pi(n^2 - m^2)]$ for odd $n + m$, 0 otherwise.]

**Solution [10]** We can write the general solution as [2]

$$u(z,t) = \sum_{n=0}^{\infty} \phi_n(z) e^{-(2n\pi/L)^2 Kt}, \qquad \phi_n(z) = \left( A_n \cos\frac{2n\pi z}{L} + B_n \sin\frac{2n\pi z}{L} \right).$$

Let us express the initial condition as [1]

$$u(z,0) = \sum_{n'=0}^{\infty} \phi_{n'}(z) = f(z), \qquad f(z) = \begin{cases} 100, & \text{for } 0 < z < L/2, \\ 0, & \text{for } L/2 < z < L. \end{cases}$$

We first integrate both sides. We have [2]

$$\sum_{n'=0}^{\infty} \phi_{n'}(z) = f(z) \quad \Rightarrow \quad \int_0^L \sum_{n'=0}^{\infty} \phi_{n'}(z)dz = \int_0^L f(z)dz \quad \Rightarrow \quad LA_0 = 100 \cdot \frac{L}{2} \quad \Rightarrow \quad A_0 = 50.$$

We next multiply $\cos(2n\pi z/L)$ ($n \geq 1$). We obtain [2]

$$\sum_{n'=0}^{\infty} \phi_{n'}(z) = f(z) \quad \Rightarrow \quad \int_0^L \cos\frac{2n\pi z}{L} \sum_{n'=0}^{\infty} \phi_{n'}(z)dz = \int_0^L \cos\frac{2n\pi z}{L} f(z)dz$$

$$\Rightarrow \quad \frac{L}{2} A_n = 100 \int_0^{L/2} \cos\frac{2n\pi z}{L} dz = 0 \quad \Rightarrow \quad A_n = 0.$$

Finally we multiply $\sin(2n\pi z/L)$ ($n \geq 1$). We obtain [2]

$$\sum_{n'=0}^{\infty} \phi_{n'}(z) = f(z) \quad \Rightarrow \quad \int_0^L \sin\frac{2n\pi z}{L} \sum_{n'=0}^{\infty} \phi_{n'}(z)dz = \int_0^L \sin\frac{2n\pi z}{L} f(z)dz$$

$$\Rightarrow \quad \frac{L}{2} B_n = 100 \int_0^{L/2} \sin\frac{2n\pi z}{L} dz = 100 \frac{L[1 - (-1)^n]}{2n\pi} \quad \Rightarrow \quad B_n = 100 \frac{1 - (-1)^n}{n\pi}.$$

We obtain [1]

$$u(z,t) = 50 + \frac{100}{\pi} \sum_{n=1}^{\infty} \left[ \frac{1 - (-1)^n}{n} \right] \sin\frac{2n\pi z}{L} e^{-(2n\pi/L)^2 Kt}.$$