MATH 454 SECTION 002

Quiz 4

February 14, 2014, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Find the steady-state solution of the heat equation $u_t = K\nabla^2 u$ in the slab $0 < z < L$, with boundary conditions $[u_z - h(u - T_0)](x, y, 0) = 0$ and $[u_z + h(u - T_1)](x, y, L) = 0$. Assume that K, h, T_0, T_1 are all positive constants.

Solution [6] In this slab problem, the solution u is independent of x, y. Furthermore the solution does not depend on t in steady state. We write the steady-state solution as [1] $U(z) = u(x, y, z, t)$. The heat equation and the boundary conditions are then written as [1]

$$U''(z) = 0, \quad 0 < z < L, \qquad U'(0) - h[U(0) - T_0] = U'(L) + h[U(L) - T_1] = 0.$$

The general solution of $U(z)$ is obtained as [1]

$$U(z) = A + Bz,$$

where constants A, B are determined by the boundary conditions. Using the boundary conditions we have

$$B - h(A - T_0) = 0, \qquad B + h(A + BL - T_1) = 0.$$

That is,

$$B = h(A - T_0), \qquad (1 + hL)B + hA - hT_1 = 0.$$

We obtain [1]

$$(1 + hL)[h(A - T_0)] + hA - hT_1 = 0 \quad \Rightarrow \quad A = \frac{(1 + hL)T_0 + T_1}{2 + hL},$$

and then [1]

$$B = h(A - T_0) \quad \Rightarrow \quad B = \frac{h(T_1 - T_0)}{2 + hL}.$$

Finally we obtain [1]

$$\begin{aligned} u(x, y, z, t) &= U(z) \\ &= \frac{(1 + hL)T_0 + T_1}{2 + hL} + \frac{h(T_1 - T_0)}{2 + hL} z \\ &= \frac{T_1(1 + hz) + T_0[1 + h(L - z)]}{2 + hL}. \end{aligned}$$