MATH 454 SECTION 002

Quiz 3

January 31, 2014, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Find the mean square error for the Fourier series of $f(x) = x^2$, $-\pi \le x \le \pi$. Then, show that $\sigma_N^2 = O(N^{-3})$ as $N \to \infty$. [*Hint:* When the Fourier series is written as $f(x) = A_0 + \sum_{n=1}^{\infty}[A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)]$, where $L = \pi$, we have $\sigma_N^2 = \frac{1}{2L}\int_{-L}^{L}[f(x) - f_N(x)]^2 dx = \frac{1}{2}\sum_{n=N+1}^{\infty}(A_n^2 + B_n^2)$. If useful, you can use the formulae $(1/L)\int_{-L}^{L} x^2 \cos(n\pi x/L)dx = 4L^2(-1)^n/(n\pi)^2$ and $(N+1)^{-3} = N^{-3}(1 - (3/N) + \cdots)$.]

Solution [6] The Fourier series is obtained as

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[A_n \cos(nx) + B_n \sin(nx)\right],$$

where $B_n = 0$ and

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)dx = \frac{4}{n^2}(-1)^n.$$

Thus, the mean square error is obtained as [2]

$$\sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty}(A_n^2 + B_n^2) = 8\sum_{n=N+1}^{\infty}\frac{1}{n^4}.$$

We note that‡ [2]

$$\sum_{n=N+1}^{\infty}\frac{1}{n^4} \le \int_{N+1}^{\infty}\frac{1}{(x-1)^4}dx.$$

The integral is calculated as

$$\int_{N+1}^{\infty}\frac{1}{(x-1)^4}dx = \int_{N}^{\infty}\frac{1}{x^4}dx = \frac{1}{3N^3}.$$

That is,

$$\sum_{n=N+1}^{\infty}\frac{1}{n^4} \le \frac{1}{3N^3}.$$

Therefore we obtain [2]

$$\sigma_N^2 = O\left(N^{-3}\right).$$

‡ The lower bound is estimated as

$$\sum_{n=N+1}^{\infty}\frac{1}{n^4} \ge \int_{N+1}^{\infty}\frac{1}{x^4}dx = \frac{1}{3(N+1)^3} = \frac{1}{3N^3}\left(1 - \frac{3}{N} + \cdots\right)$$

Therefore we obtain

$$\sum_{n=N+1}^{\infty}\frac{1}{n^4} = \frac{1}{3N^3} + O\left(\frac{1}{N^4}\right).$$