MATH 454 SECTION 002

Quiz 1

January 17, 2014, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Find the separated solutions of $u_{xx} + yu_y + u = 0$. [*Hint:* Classify the solution into three cases so that you always have nonnegative numbers inside the square root and so that i doesn't appear explicitly. Also you can use the fact that the solution to $xf'(x) + af(x) = 0$ (a is a constant) is $f(x) = C/|x|^a = C'/x^a$, where C, C' are constants.]

Solution [6] We substitute $X(x)Y(y)$ for $u(x,y)$ in the equation and obtain $X''Y + yXY' + XY = 0$. By introducing the separation constant λ, we have [2] $\frac{X''}{X} = \lambda$, $y\frac{Y'}{Y} + 1 = -\lambda$, or

$$X'' - \lambda X = 0, \quad yY' + (1+\lambda)Y = 0.$$

The first equation has two linearly independent solutions $e^{\pm\sqrt{\lambda}x}$ for $\lambda \neq 0$, and $1, x$ for $\lambda = 0$. Therefore the general solution is written as [2]

$$X = \begin{cases} A_1 + A_2 x & \lambda = 0, \\ A_1 e^{\sqrt{\lambda}x} + A_2 e^{-\sqrt{\lambda}x} & \lambda \neq 0, \end{cases}$$

where A_1, A_2 are constants. The second equation is solved as

$$Y = \frac{C}{|y|^{1+\lambda}},$$

where C is a constant. We note that $A_1 e^{\sqrt{\lambda}x} + A_2 e^{-\sqrt{\lambda}x} = A_1 e^{i\sqrt{|\lambda|}x} + A_2 e^{-i\sqrt{|\lambda|}x} = (A_1 + A_2)\cos(\sqrt{|\lambda|}x) + i(A_1 - A_2)\sin(\sqrt{|\lambda|}x)$ for $\lambda < 0$. Therefore we obtain [2]

$$u(x,y) = \begin{cases} \left(A_1 e^{\sqrt{\lambda}x} + A_2 e^{-\sqrt{\lambda}x}\right)\left(1/|y|^{1+\lambda}\right) & \text{for } \lambda > 0, \\ (A_1 + A_2 x)\,(1/|y|) & \text{for } \lambda = 0, \\ \left(A_1 \cos(\sqrt{|\lambda|}x) + A_2 \sin(\sqrt{|\lambda|}x)\right)\left(1/|y|^{1+\lambda}\right) & \text{for } \lambda < 0, \end{cases}$$

where A_1, A_2 are constants.

Alternative Solution For example we can also choose λ as $\frac{X''}{X} + 1 = -\lambda$, $y\frac{Y'}{Y} = \lambda$, or

$$X'' + (1+\lambda)X = 0, \quad yY' - \lambda Y = 0.$$

Then we obtain

$$u(x,y) = \begin{cases} \left(A_1 e^{\sqrt{|\lambda|-1}x} + A_2 e^{-\sqrt{|\lambda|-1}x}\right)|y|^{\lambda} & \text{for } \lambda < -1, \\ (A_1 + A_2 x)\,(1/|y|) & \text{for } \lambda = -1, \\ \left(A_1 \cos(\sqrt{\lambda+1}x) + A_2 \sin(\sqrt{\lambda+1}x)\right)|y|^{\lambda} & \text{for } \lambda > -1. \end{cases}$$