Chapter 7
Green's functions

What we know

We have already seen Green's functions in Chapter 5. For example the heat equation (5.1), $u_t = Ku_{xx}$ ($x \in (-\infty,\infty)$, $t > 0$) with the initial condition $u(x,0) = f(x)$, is solved as

$$u(x,t) = \int_{-\infty}^{\infty} G(x,x';t) f(x') dx', \tag{7.1}$$

where $G(x,x';t)$ is the heat kernel given in (5.4):

$$G(x,x';t) = \frac{1}{\sqrt{4\pi Kt}} e^{-(x-x')^2/(4Kt)}. \tag{7.2}$$

Thus the linear partial differential equation is solved in terms of an integral transform.

Let us consider the wave equation (5.7), $u_{tt} = c^2 u_{xx}$ ($x \in (-\infty,\infty)$, $t > 0$) with the initial conditions $u(x,0) = f_1(x)$, $u_t(x,0) = f_2(x)$. As shown in (5.9), similarly the solution is written as

$$u(x,t) = \int_{-\infty}^{\infty} G^{(1)}(x,x';t) f_1(x') dx' + \int_{-\infty}^{\infty} G^{(2)}(x,x';t) f_2(x') dx',$$

where

$$G^{(1)}(x,x';t) = \frac{1}{2} \left[\delta(x - x' + ct) + \delta(x - x' - ct) \right],$$
$$G^{(2)}(x,x';t) = \frac{1}{4c} \left[\mathrm{sgn}(x - x' + ct) - \mathrm{sgn}(x - x' - ct) \right].$$

Winter 2014 Math 454 Sec 2
Boundary Value Problems for Partial Differential Equations

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The Green's function for the heat equation [1]

If the Green's function is known, we can write down the solution as integral forms. Hereafter we will focus on the Green's function for the heat equation.

Case 1 $(-\infty < x < \infty)$

Let us consider

$$\begin{cases} u_t - Ku_{xx} = h(x,t), & 0 < t < T, \quad -\infty < x < \infty, \\ u = f(x), & t = 0, \quad -\infty < x < \infty. \end{cases}$$

We write

$$u(x,t) = v(x,t) + w(x,t),$$

and split the equation into two equations:

$$\begin{cases} v_t - Kv_{xx} = 0, & 0 < t < T, \quad -\infty < x < \infty, \\ v = f(x), & t = 0, \quad -\infty < x < \infty, \end{cases}$$

and

$$\begin{cases} w_t - Kw_{xx} = h(x,t), & 0 < t < T, \quad -\infty < x < \infty, \\ w = 0, & t = 0, \quad -\infty < x < \infty. \end{cases}$$

Let us look at the equation for w. Using the Fourier transform $\tilde{w}(\mu,t)$, we have

$$\tilde{w}_t + \mu^2 K\tilde{w} = \tilde{h}(\mu,t), \qquad \tilde{w}(\mu,0) = 0.$$

Noting $\frac{d}{dt}\left[\tilde{w}e^{\mu^2 Kt}\right] = \tilde{w}_t e^{\mu^2 Kt} + \mu^2 K\tilde{w}e^{\mu^2 Kt}$, we obtain

$$\tilde{w} = \int_0^t e^{-\mu^2 K(t-s)}\tilde{h}(\mu,s)ds.$$

We obtain w as

[1] This section corresponds to §8.4 of the textbook.

$$\begin{aligned}
w(x,t) &= \int_{-\infty}^{\infty}\left[\int_0^t e^{-\mu^2K(t-s)}\tilde{h}(\mu,s)ds\right]e^{i\mu x}d\mu \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t\int_{-\infty}^{\infty} e^{-\mu^2K(t-s)}h(x',s)e^{i\mu(x-x')}dx'dsd\mu \\
&= \frac{1}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-K(t-s)\left(\mu-i\frac{x-x'}{2K(t-s)}\right)^2\right]e^{-(x-x')^2/[4K(t-s)]}h(x',s)d\mu dx'ds \\
&= \int_0^t\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi K(t-s)}}e^{-(x-x')^2/[4K(t-s)]}h(x',s)dx'ds.
\end{aligned}$$

Therefore we obtain

$$u(x,t) = \int_{-\infty}^{\infty}G(x,x';t)f(x')dx' + \int_0^t\int_{-\infty}^{\infty}G(x,x';t-s)h(x',s)dx'ds.$$

Case 2 $(0 < x < \infty)$

Recall that (5.5), $u_t = Ku_{xx}$ $(t > 0, x \in (0,\infty))$ with the Dirichlet boundary condition $u(0,t) = 0$ and initial condition $u(x,0) = f(x)$, is solved as

$$u(x,t) = \int_0^{\infty}G_D(x,x';t)f(x')dx',$$

where

$$G_D(x,x';t) = G(x,x';t) - G(x,-x';t),$$

and for the Neumann boundary condition $u_x(0,t) = 0$ we have

$$u(x,t) = \int_0^{\infty}G_N(x,x';t)f(x')dx',$$

where

$$G_N(x,x';t) = G(x,x';t) + G(x,-x';t).$$

Let us consider

$$\begin{cases} u_t - Ku_{xx} = h(x,t), & 0 < t < T, \quad 0 < x < \infty, \\ u = 0, & 0 < t < T, \quad x = 0, \\ u = 0, & t = 0, \quad 0 < x < \infty. \end{cases}$$

We extend h as

$$h_O(x,t) = \begin{cases} h(x,t), & x > 0, \\ 0, & x = 0, \\ -h(-x,t), & x < 0. \end{cases}$$

Then we have

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,x',t-s) h_O(x',s) dx' ds.$$

We obtain

$$u(x,t) = \int_0^t \int_0^{\infty} G_D(x,x',t-s) h(x',s) dx' ds.$$

Next we consider

$$\begin{cases} u_t - K u_{xx} = h(x,t), & 0 < t < T, \quad 0 < x < \infty, \\ u = 0, & 0 < t < T, \quad x = 0, \\ u = f(x), & t = 0, \quad 0 < x < \infty. \end{cases}$$

The solution is obtained as

$$u(x,t) = \int_0^{\infty} G_D(x,x';t) f(x') dx' + \int_0^t \int_0^{\infty} G_D(x,x',t-s) h(x',s) dx' ds.$$

In the case of the Neumann boundary condition $u_x = 0$, we obtain

$$u(x,t) = \int_0^{\infty} G_N(x,x';t) f(x') dx' + \int_0^t \int_0^{\infty} G_N(x,x',t-s) h(x',s) dx' ds.$$

Case 3 $(0 < x < L)$

Let us solve

$$\begin{cases} u_t - K u_{xx} = 0, & 0 < t < T, \quad 0 < x < L, \\ u = 0, & 0 < t < T, \quad x = 0, \\ u = 0, & 0 < t < T, \quad x = L, \\ u = f(x), & t = 0, \quad 0 < x < L. \end{cases}$$

We have solved this equation using separation of variables, and obtained (cf., Chapter 2)

$$u(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L f(x') \sin \frac{n\pi x'}{L} dx' \right] \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 K t}. \tag{7.3}$$

Therefore we can read off the Green's function $G_L(x,x';t)$ as

$$G_L(x,x';t) = \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi x'}{L} e^{-(n\pi/L)^2 K t}.$$

We will find another expression of $u(x,t)$ using the Fourier transform.

We extend $f(x)$ as an odd $2L$-periodic function by setting

$$f_O(x) = \begin{cases} f(x-2mL), & 2mL < x < (2m+1)L, \\ 0, & x = 2mL, \quad (2m+1)L, \quad (2m+2)L, \\ -f(-x+(2m+2)L), & (2m+1)L < x < (2m+2)L, \end{cases}$$

where $m = 0, \pm 1, \pm 2, \ldots$. Note that $f_O(x+2L) = f_O(x)$ for all x. Then we have

$$u(x,t) = \int_{-\infty}^{\infty} G(x,x';t) f_O(x') dx' = \sum_{m=-\infty}^{\infty} \left\{ \int_{2mL}^{(2m+1)L} + \int_{(2m+1)L}^{(2m+2)L} \right\} G(x,x';t) f_O(x') dx'.$$

We obtain

$$u(x,t) = \int_0^L G_L(x,x';t) f(x') dx', \tag{7.4}$$

where

$$G_L(x,x';t) = \sum_{m=-\infty}^{\infty} \left[G(x,x'+2mL;t) - G(x,-x'+(2m+2)L;t) \right].$$

(7.4) is another expression of (7.3)

We can similarly solve

$$\begin{cases} u_t - K u_{xx} = h(x,t), & 0 < t < T, \quad 0 < x < L, \\ u = 0, & 0 < t < T, \quad x = 0, \\ u = 0, & 0 < t < T, \quad x = L, \\ u = f(x), & t = 0, \quad 0 < x < L. \end{cases}$$

We extend $h(x,t)$ as an odd $2L$-periodic function by setting

$$h_O(x,t) = \begin{cases} h(x-2mL,t), & 2mL < x < (2m+1)L, \\ 0, & x = 2mL, \quad (2m+1)L, \quad (2m+2)L, \\ -h(-x+(2m+2)L,t), & (2m+1)L < x < (2m+2)L, \end{cases}$$

Note that $h_O(x+2L,t) = h_O(x,t)$ for all x. Then we have

$$u(x,t) = \int_0^L G_L(x,x',t) f(x') dx' + \int_0^t \int_0^L G_L(x,x',t-s) h(x',s) dx' ds.$$

One dimension [2]

Let us consider the following ordinary differential equation.

$$\begin{cases} y'' = -f, & 0 < x < L, \\ y = 0, & x = 0, L, \end{cases}$$

where $f(x)$, $0 < x < L$, is a piecewise smooth function.

We integrate both sides and obtain $y'(x) = y'(0) - \int_0^x f(x')dx'$. By integrating one more time, we obtain

$$\begin{aligned} y(x) &= y(0) + y'(0)x - \int_0^x \int_0^{x'} f(x'')dx''dx' \\ &= y'(0)x - \int_0^x \int_{x''}^x f(x'')dx'dx'' \\ &= y'(0)x - \int_0^x (x - x'')f(x'')dx'', \end{aligned}$$

where we used $\int_0^x dx' \int_0^{x'} dx'' \cdots = \int_0^x dx'' \int_{x''}^x dx' \ldots$. We note that

$$y(L) = y'(0)L - \int_0^L (L - x')f(x')dx' = 0.$$

Hence,

$$\begin{aligned} y(x) &= \frac{x}{L} \int_0^L (L - x')f(x')dx' - \int_0^x (x - x')f(x')dx' \\ &= \int_0^x \left[\frac{x}{L}(L - x') - (x - x') \right] f(x')dx' + \frac{x}{L} \int_x^L (L - x')f(x')dx' \\ &= \int_0^x \frac{x'}{L}(L - x)f(x')dx' + \int_x^L \frac{x}{L}(L - x')f(x')dx'. \end{aligned}$$

Therefore we can write

$$y(x) = \int_0^L G(x, x')f(x')dx', \tag{7.5}$$

where

$$G(x, x') = \begin{cases} \dfrac{x'(L - x)}{L}, & 0 \le x' \le x, \\ \dfrac{x(L - x')}{L}, & x \le x' \le L. \end{cases} \tag{7.6}$$

We note that the Green's function $G(x, x')$ depends only on the equation and boundary conditions, and is independent of $f(x)$.

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[2] This section corresponds to §8.1 of the textbook.

We note that the Green's function $G(x,x')$ is the solution to

$$\begin{cases} \partial_x^2 G = -\delta(x-x'), & 0<x<L, \\ G=0, & x=0,L. \end{cases} \tag{7.7}$$

Let us first confirm that if $G(x,x')$ is the solution to (7.7), then $y(x)$ is given by (7.5). From (7.5) we have

$$y''(x) = \int_0^L \partial_x^2 G(x,x') f(x') dx' = -\int_0^L \delta(x-x') f(x') dx' = -f(x).$$

Moreover $y(0) = \int_0^L G(0,x') f(x') dx' = \int_0^L (0) f(x') dx' = 0$ and similarly $y(L) = 0$.

Let us integrate the equation from $x = x'-0$ to $x'+0$. We have

$$G_x(x'+0,x') - G_x(x'-0,x') = -\int_{x'-0}^{x'+0} \delta(x-x') dx' = -1.$$

Thus G_x has a jump at $x = x'$. If we integrate (7.7) from 0 to x, we obtain

$$\partial_x G(x,x') = G_x(0,x') - \theta(x-x'),$$

where $\theta(x-x')$ is the step function: $\theta(x) = 1$ for $x>0$, $=1/2$ for $x=0$, and $=0$ for $x<0$. By integrating the above equation from $x = x'-0$ to $x'+0$, we obtain

$$G(x'+0,x') - G(x'-0,x') = \int_{x'-0}^{x'+0} G_x(0,x') dx - \int_{x'-0}^{x'+0} \theta(x-x') dx = 0.$$

Hence G is continuous at $x = x'$.

Remark 1. The Green's function (7.2) is the solution to

$$\begin{cases} G_t = KG_{xx}, & t>0, \quad x \in (-\infty,\infty), \\ G = \delta(x-x'), & t=0, \quad x,x' \in (-\infty,\infty). \end{cases}$$

Note that $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mu x} d\mu$.

We can solve (7.7) and get (7.6) just like we derived (7.5). Here let us solve (7.7) by using the Sturm-Louville eigenproblem.

Let us consider

$$\phi_n''(x) + \lambda_n \phi(x) = 0, \qquad \phi_n(0) = \phi_n(L) = 0.$$

We obtain

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1,2,\ldots.$$

We have almost always chosen the coefficient in ϕ_n to be 1, but here we choose $\sqrt{2/L}$ noticing that $\int_0^L \sin(n\pi x/L)^2 dx = L/2$. Thus in this case

$$\int_0^L \phi_n(x)^2 dx = 1.$$

Or we can write

$$\langle \phi_n, \phi_m \rangle = \delta_{nm}, \qquad \|\phi_n\| = 1.$$

We expanded the functions $v(z,t)$, $R(z,t)$, and $F(z)$ with the Sturm-Liouville eigenfunctions when we solved (2.19) in Chapter 2. Similarly we write the Green's function as

$$G(x,x') = \sum_{n=1}^{\infty} A_n \phi_n(x).$$

We have

$$G_{xx}(x,x') = \sum_{n=1}^{\infty} A_n \phi_n''(x) = -\sum_{n=1}^{\infty} A_n \lambda_n \phi_n(x) = -\delta(x-x').$$

We multiply $\phi_m(x)$ and integrate both sides.

$$\int_0^L \sum_{n=1}^{\infty} A_n \lambda_n \phi_n(x) \phi_m(x) dx = \int_0^L \delta(x-x') \phi_m(x) dx.$$

$$\text{LHS} = \sum_{n=1}^{\infty} A_n \lambda_n \delta_{nm} = A_m \lambda_m, \qquad \text{RHS} = \phi_m(x').$$

Hence,

$$A_n = \frac{\phi_n(x')}{\lambda_n}.$$

We obtain

$$G(x,x') = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(x')}{\lambda_n}. \tag{7.8}$$

This (7.8) is another expression of (7.6). The series in (7.8) converges uniformly for $x, x' \in [0, L]$.

The Green's function $G(x,x')$ has the following properties.

1. $G_{xx} = 0$ except when $x = x'$ (homogeneous equation).
2. $G(0,x') = 0$ and $G(L,x') = 0$ (boundary conditions).
3. $G(x'+0,x') - G(x'-0,x') = 0$ (continuity).
4. $G_x(x'+0,x') - G_x(x'-0,x') = -1$ (jump).
5. $G(x,x') = G(x',x)$ (reciprocity).

The Green's function $G(x,x')$ is continuous but G_x has a jump at $x = x'$.

The conditions 1 through 4 uniquely determine $G(x,x')$. Indeed, by Condition 1 we can write $G(x,x') = Ax+B$ for $x > x'$ and $Cx+D$ for $x < x'$. Using Condition 2 we have $G(x,x') = A(x-L)$ for $x > x'$ and Cx for $x < x'$. Condition 3 implies $A(x'-L) = Cx'$, and Condition 4 implies $A-C = -1$. Thus we uniquely obtain (7.6).

Condition 5 is called the reciprocity relation. It means that G at x for the source at x' is the same as G at x' for the source at x.

Theorem 1 (Reciprocity). *We consider the Green's function $G(x,x')$ in (7.7). For $x,x' \in [0,L]$, we have*

$$G(x,x') = G(x',x).$$

Proof. We consider two sources:

$$G_{xx}(x,x_1) = -\delta(x-x_1), \qquad G_{xx}(x,x_2) = -\delta(x-x_2),$$

where $x,x_1,x_2 \in [0,L]$, and $G = 0$ for $x = 0,L$. We multiply the first equation by $G(x,x_2)$ and the second equation by $G(x,x_1)$ and integrate two equations:

$$\int_0^L G(x,x_2)G_{xx}(x,x_1)dx = -\int_0^L G(x,x_2)\delta(x-x_1)dx,$$
$$\int_0^L G(x,x_1)G_{xx}(x,x_2)dx = -\int_0^L G(x,x_1)\delta(x-x_2)dx.$$

We note that

$$\begin{aligned}
\int_0^L G(x,x_2)G_{xx}(x,x_1)dx &= G(x,x_2)G_x(x,x_1)\Big|_0^L - \int_0^L G_x(x,x_2)G_x(x,x_1)dx \\
&= -\int_0^L G_x(x,x_2)G_x(x,x_1)dx, \\
\int_0^L G(x,x_1)G_{xx}(x,x_2)dx &= G(x,x_1)G_x(x,x_2)\Big|_0^L - \int_0^L G_x(x,x_1)G_x(x,x_2)dx \\
&= -\int_0^L G_x(x,x_1)G_x(x,x_2)dx.
\end{aligned}$$

Therefore we obtain

$$\int_0^L G(x,x_2)\delta(x-x_1)dx = \int_0^L G(x,x_1)\delta(x-x_2)dx.$$

This implies $G(x_1,x_2) = G(x_2,x_1)$ and completes the proof. □