Chapter 5
Fourier transforms

Basic properties of the Fourier transform [1]

Let us recall the complex form of Fourier series in Chapter 1. A function $f(x)$, $-L < x < L$, is expressed as

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}, \quad \alpha_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L} dx.$$

We define

$$\mu_n = \frac{n\pi}{L} \quad (n = 0, \pm 1, \pm 2, \cdots), \qquad \Delta\mu_n = \mu_{n+1} - \mu_n = \frac{\pi}{L}.$$

We obtain

$$f(x) = \sum_{n=-\infty}^{\infty} \tilde{f}_L(\mu_n) e^{i\mu_n x} \Delta\mu_n,$$

where

$$\tilde{f}_L(\mu) = \frac{1}{2\pi}\int_{-L}^{L} f(x) e^{-i\mu x} dx.$$

Now we let L go to infinity, $L \to \infty$.

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(\mu) e^{i\mu x} d\mu, \qquad \tilde{f}(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-i\mu x} dx.$$

The above $\tilde{f}(\mu)$ is called the Fourier transform of $f(x)$.

Let us also recall Parseval's theorem in complex form:

$$\int_{-L}^{L} f(x)^2 dx = 2L \sum_{n=-\infty}^{\infty} |\alpha_n|^2.$$

We note that

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Boundary Value Problems for Partial Differential Equations

Manabu Machida (University of Michigan)

[1] This section corresponds to §5.1 of the textbook.

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty}|\alpha_n|^2 &= \left(\frac{1}{2L}\right)^2 \sum_{n=-\infty}^{\infty}\left|\int_{-L}^{L} f(x)e^{-in\pi x/L}dx\right|^2 \\
&= \left(\frac{1}{2L}\right)^2 \sum_{n=-\infty}^{\infty}\left|\int_{-L}^{L} \sum_{m=-\infty}^{\infty}\tilde{f}_L(\mu_m)e^{i\mu_m x}\Delta\mu_m e^{-in\pi x/L}dx\right|^2 \\
&= \sum_{n=-\infty}^{\infty}\left|\tilde{f}_L(\mu_n)\Delta\mu_n\right|^2 \\
&= \frac{\pi}{L}\sum_{n=-\infty}^{\infty}\left|\tilde{f}_L(\mu_n)\right|^2\Delta\mu_n,
\end{aligned}
$$

where we used orthogonality relations $\int_{-L}^{L} e^{i(m-n)\pi x/L}dx = 2L\delta_{mn}$. Thus Parseval's theorem for Fourier transforms is obtained as

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)^2 dx = \int_{-\infty}^{\infty}|\tilde{f}(\mu)|^2 d\mu .
$$

Gaussian integral

Consider a Gaussian $f(x) = e^{-a(x-b)^2}$, $a > 0$.

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ax^2}e^{-ay^2}dxdx} &= \sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-a\rho^2}\rho d\rho d\varphi} \\
&= \sqrt{2\pi \left.\frac{e^{-a\rho^2}}{-2a}\right|_0^{\infty}} = \sqrt{\frac{\pi}{a}}.
\end{aligned}
$$

This is the famous Gaussian integral. The result $\int_{-\infty}^{\infty} e^{-a(x-b)^2}dx = \sqrt{\pi/a}$ is true even when $b \in \mathbb{C}$.

Example 1. We obtain the Fourier transform of the Gaussian $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-m)^2/2\sigma^2}$ (m is the mean and σ is the standard deviation) as

$$
\begin{aligned}
\tilde{f}(\mu) &= \frac{1}{2\pi\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\sigma^2}x^2 + \left(\frac{m}{\sigma^2} - i\mu\right)x - \frac{m^2}{2\sigma^2}\right]dx \\
&= \frac{1}{2\pi\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\left[x-(m-i\mu\sigma^2)\right]^2/(2\sigma^2) - im\mu - \mu^2\sigma^2/2}dx \\
&= \frac{1}{2\pi}e^{-im\mu}e^{-\mu^2\sigma^2/2}.
\end{aligned}
$$

Fourier cosine and sine transforms

Let $f(x)$ be defined for $x > 0$. We extend f as an even function f_E ($f_E(-x) = f_E(x)$) or as an odd function f_O ($f_O(-x) = -f_O(x)$).

$$f(x) = \int_0^\infty \tilde{f}_c(\mu)\cos(\mu x)d\mu, \quad \tilde{f}_c(\mu) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\mu x)dx,$$
$$f(x) = \int_0^\infty \tilde{f}_s(\mu)\sin(\mu x)d\mu, \quad \tilde{f}_s(\mu) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\mu x)dx.$$

The delta function

Let us consider the Fourier transform of 1, i.e., $f(x) = 1$. Interestingly,

$$\tilde{f}(\mu) = \delta(\mu) = \begin{cases} \infty, & \mu = 0, \\ 0, & \mu \neq 0. \end{cases}$$

This $\delta(\mu)$ is called Dirac's delta function. Indeed $\delta(\mu)$ is a generalized function. We can consider the delta function through Gaussian. We write $f(x) = 1 = \lim_{\sigma\to\infty} e^{-x^2/2\sigma^2}$. Then,

$$\begin{aligned}\tilde{f}(\mu) &= \lim_{\sigma\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}e^{-i\mu x}dx \\ &= \lim_{\sigma\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x+i\mu\sigma^2)^2-\mu^2\sigma^2/2}dx \\ &= \lim_{\sigma\to\infty}\frac{1}{\sqrt{2\pi}(1/\sigma)}e^{-\frac{\mu^2}{2(1/\sigma)^2}}.\end{aligned}$$

The right-hand side is an extremely sharp Gaussian.

For some function $g(x)$,

$$\int_{-\infty}^{\infty} g(x)\delta(x-a)dx = g(a).$$

This property is compared to $\sum_m g_m\delta_{nm} = g_n$ with Kronecker's delta δ_{nm}.

Furthermore we have

$$\int_{-\infty}^{\infty} g(x)\delta(bx-a)dx = \int_{-\infty}^{\infty} g(x)\frac{1}{|b|}\delta\left(x-\frac{a}{b}\right)dx = \frac{g(a/b)}{|b|},$$

where a, b are constants. This implies $\delta(-x) = \delta(x)$.

The heat equation for an infinite rod [2]

We will solve the following heat equation.

$$\begin{cases} u_t = Ku_{xx}, & t > 0, \quad -\infty < x < \infty, \\ u(x,0) = f(x), & -\infty < x < \infty. \end{cases} \tag{5.1}$$

Let us use the Fourier transform.

$$u(x,t) = \int_{-\infty}^{\infty} \tilde{u}(\mu,t)e^{i\mu x}d\mu, \quad \tilde{u}(\mu,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-i\mu x}dx.$$

We note that

$$u_t(x,t) = \int_{-\infty}^{\infty} \tilde{u}_t(\mu,t)e^{i\mu x}d\mu, \qquad u_{xx}(x,t) = \int_{-\infty}^{\infty} \tilde{u}(\mu,t)(-\mu^2)e^{i\mu x}d\mu.$$

Hence,

$$\int_{-\infty}^{\infty} \tilde{u}_t(\mu,t)e^{i\mu x}d\mu = K\int_{-\infty}^{\infty} \tilde{u}(\mu,t)(-\mu^2)e^{i\mu x}d\mu.$$

We multiply e^{-ikx} and integrate both sides over x:

$$\int_{-\infty}^{\infty} dxe^{-ikx}\int_{-\infty}^{\infty} \tilde{u}_t(\mu,t)e^{i\mu x}d\mu = K\int_{-\infty}^{\infty} dxe^{-ikx}\int_{-\infty}^{\infty} \tilde{u}(\mu,t)(-\mu^2)e^{i\mu x}d\mu.$$

Since $\int_{-\infty}^{\infty} e^{i(\mu-k)x}dx = 2\pi\delta(\mu-k)$, we obtain

$$\partial_t\tilde{u}(k,t) + k^2K\tilde{u}(k,t) = 0. \tag{5.2}$$

From the initial condition $f(x) = \int_{-\infty}^{\infty} \tilde{u}(\mu,0)e^{i\mu x}d\mu$, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} dxe^{-ikx}\int_{-\infty}^{\infty} \tilde{u}(\mu,0)e^{i\mu x}d\mu.$$

Thus

$$\tilde{u}(k,0) = \tilde{f}(k). \tag{5.3}$$

By solving (5.2) and (5.3), we obtain

$$\tilde{u}(k,t) = \tilde{f}(k)e^{-k^2Kt}.$$

We obtain

$$u(x,t) = \int_{-\infty}^{\infty} \tilde{f}(\mu)e^{-\mu^2Kt}e^{i\mu x}d\mu.$$

Furthermore,

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[2] This section corresponds to §5.2 of the textbook.

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\left[\int_{-\infty}^{\infty} e^{-i\mu x'} e^{i\mu x} e^{-\mu^2 Kt} d\mu\right] dx'$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\left[\int_{-\infty}^{\infty} \exp\left[-Kt\left(\mu - \frac{i(x-x')}{2Kt}\right)^2\right] e^{-(x-x')^2/4Kt} d\mu\right] dx'.$$

Thus we obtain

$$u(x,t) = \frac{1}{\sqrt{4\pi Kt}}\int_{-\infty}^{\infty} e^{-(x-x')^2/4Kt} f(x') dx'. \tag{5.4}$$

The function $\frac{e^{-(x-x')^2/4Kt}}{\sqrt{4\pi Kt}}$ is called the Gauss-Weierstrass kernel, the heat kernel, the fundamental solution to the heat equation, or the Green's function of the heat equation.

The method of images

We will consider the heat equation in the half space.

Dirichlet Boundary Condition

$$\begin{cases} u_t = Ku_{xx}, & t > 0, \quad 0 < x < \infty, \\ u(0,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < \infty. \end{cases} \tag{5.5}$$

We extend $f(x)$ as

$$f_O(x) = \begin{cases} f(x), & x > 0, \\ 0, & x = 0, \\ -f(-x), & x < 0. \end{cases}$$

We have

$$\begin{cases} u_t = Ku_{xx}, & t > 0, \quad -\infty < x < \infty, \\ u(x,0) = f_O(x), & -\infty < x < \infty. \end{cases} \tag{5.6}$$

Note that the heat equation is invariant under the change $x \to -x$, and $u(x,t)$ and $-u(-x,t)$ satisfy the same equation and the same initial condition. Hence $-u(-x,t) = u(x,t)$. In particular $-u(0,t) = u(0,t)$, which implies $u(0,t) = 0$. Thus we see that the right half $(0 < x < \infty)$ of (5.6) is equivalent to (5.5).

For $f_O(x)$ defined on $-\infty < x < \infty$, we have

$$u(x,t)=\frac{1}{\sqrt{4\pi Kt}}\int_{-\infty}^{\infty}f_O(x')e^{-(x-x')^2/4Kt}dx'=\left(\int_{-\infty}^{0}+\int_{0}^{\infty}\right)\frac{e^{-(x-x')^2/4Kt}}{\sqrt{4\pi Kt}}f_O(x')dx'.$$

Therefore,

$$u(x,t)=\int_0^\infty\frac{e^{-(x-x')^2/4Kt}-e^{-(x+x')^2/4Kt}}{\sqrt{4\pi Kt}}f(x')dx'.$$

Neumann Boundary Condition

$$\begin{cases} u_t=Ku_{xx}, & t>0, \quad 0<x<\infty, \\ u_x(0,t)=0, & t>0, \\ u(x,0)=f(x), & 0<x<\infty. \end{cases}$$

We extend $f(x)$ as

$$f_E(x)=\begin{cases} f(x), & x>0, \\ f(-x), & x<0. \end{cases}$$

We have

$$\begin{cases} u_t=Ku_{xx}, & t>0, \quad -\infty<x<\infty, \\ u(x,0)=f_E(x), & -\infty<x<\infty. \end{cases}$$

For $f_E(x)$ defined on $-\infty<x<\infty$, we have

$$u(x,t)=\frac{1}{\sqrt{4\pi Kt}}\int_{-\infty}^{\infty}f_E(x')e^{-(x-x')^2/4Kt}dx'=\left(\int_{-\infty}^{0}+\int_{0}^{\infty}\right)\frac{e^{-(x-x')^2/4Kt}}{\sqrt{4\pi Kt}}f_E(x')dx'.$$

Thus,

$$u(x,t)=\int_0^\infty\frac{e^{-(x-x')^2/4Kt}+e^{-(x+x')^2/4Kt}}{\sqrt{4\pi Kt}}f(x')dx'.$$

d'Alembert's formula [3]

Let us consider the following wave equation.

$$\begin{cases} u_{tt}=c^2u_{xx}, & t>0, \quad -\infty<x<\infty, \\ u(x,0)=f_1(x), & -\infty<x<\infty, \\ u_t(x,0)=f_2(x), & -\infty<x<\infty. \end{cases} \tag{5.7}$$

We use the Fourier transform:

[3] This section corresponds to §5.3 of the textbook.

$$\begin{cases} u(x,t) = \int_{-\infty}^{\infty} \tilde{u}(\mu,t)e^{i\mu x}d\mu, & \tilde{u}(\mu,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-i\mu x}dx, \\ f_1(x) = \int_{-\infty}^{\infty} \tilde{f}_1(\mu)e^{i\mu x}d\mu, & \tilde{f}_1(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(x)e^{-i\mu x}dx, \\ f_2(x) = \int_{-\infty}^{\infty} \tilde{f}_2(\mu)e^{i\mu x}d\mu, & \tilde{f}_2(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_2(x)e^{-i\mu x}dx. \end{cases}$$

The wave equation reduces to

$$\begin{cases} \tilde{u}_{tt} + c^2\mu^2\tilde{u} = 0, & t > 0, \quad -\infty < \mu < \infty, \\ \tilde{u}(\mu,0) = \tilde{f}_1(\mu), & -\infty < \mu < \infty, \\ \tilde{u}_t(\mu,0) = \tilde{f}_2(\mu), & -\infty < \mu < \infty. \end{cases}$$

Using coefficients $A(\mu), B(\mu)$, we obtain $\tilde{u}$ as

$$\tilde{u}(\mu,t) = A(\mu)\cos(\mu ct) + B(\mu)\sin(\mu ct).$$

Using the initial conditions we find

$$A(\mu) = \tilde{f}_1(\mu), \qquad B(\mu) = \frac{\tilde{f}_2(\mu)}{\mu c}.$$

Therefore,

$$u(x,t) = \int_{-\infty}^{\infty} \left[\tilde{f}_1(\mu)\cos(\mu ct) + \tilde{f}_2(\mu)\frac{\sin(\mu ct)}{\mu c}\right] e^{i\mu x}d\mu.$$

This can be written as

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \tilde{f}_1(\mu)\frac{e^{i\mu(x+ct)} + e^{i\mu(x-ct)}}{2}d\mu + \int_{-\infty}^{\infty} \tilde{f}_2(\mu)\frac{e^{i\mu(x+ct)} - e^{i\mu(x-ct)}}{2i\mu c}d\mu \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \tilde{f}_1(\mu)\left[e^{i\mu(x+ct)} + e^{i\mu(x-ct)}\right]d\mu + \frac{1}{2c}\int_{-\infty}^{\infty} \tilde{f}_2(\mu)\int_{x-ct}^{x+ct} e^{i\mu y}dyd\mu. \end{aligned}$$

Thus we obtain d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left[f_1(x+ct) + f_1(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(y)dy. \tag{5.8}$$

We note that d'Alembert's formula can be expressed as

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{\delta(x+ct-x') + \delta(x-ct-x')}{2}f_1(x')dx' \\ &\quad + \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x+ct-x') - \operatorname{sgn}(x-ct-x')}{4c}f_2(x')dx'. \end{aligned} \tag{5.9}$$

Here $\delta(x)$ is the Dirac delta function, and the sign function is defined as $\mathrm{sgn}(x) = 1$ $(x > 0)$, $= 0$ $(x = 0)$, $= -1$ $(x < 0)$. We can write u as

$$u(x,t) = g_1(x - ct) + g_2(x + ct), \tag{5.10}$$

where

$$g_1(y) = \int_{-\infty}^{\infty} \left[\frac{\delta(y - x')}{2} f_1(x') - \frac{\mathrm{sgn}(y - x')}{4c} f_2(x') \right] dx',$$
$$g_2(y) = \int_{-\infty}^{\infty} \left[\frac{\delta(y - x')}{2} f_1(x') + \frac{\mathrm{sgn}(y - x')}{4c} f_2(x') \right] dx'.$$

We can readily check that u of the form (5.10) satisfies the wave equation.

Example 2. Let us solve the following wave equation.

$$\begin{cases} u_{tt} = u_{xx}, & t > 0, \quad -\infty < x < \infty, \\ u(x,0) = 0, & -\infty < x < \infty, \\ u_t(x,0) = x, & -\infty < x < \infty. \end{cases}$$

The solution is

$$u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} y dy = xt.$$

The heat equation and wave equation

The heat equation is parabolic and the wave equation is hyperbolic (Chapter 1). Let us compare the solutions to these equations.

We set $f(x) = e^{-x^2}$ in the heat equation (5.1):

$$\begin{cases} u_t = K u_{xx}, & t > 0, \quad -\infty < x < \infty, \\ u(x,0) = e^{-x^2}, & -\infty < x < \infty. \end{cases}$$

By (5.4), the solution is obtained as

$$u(x,t) = \frac{1}{\sqrt{4\pi Kt}} \int_{-\infty}^{\infty} e^{-(x-x')^2/4Kt} e^{-x'^2} dx' = \frac{1}{\sqrt{1 + 4Kt}} e^{-x^2/(1+4Kt)}. \tag{5.11}$$

Next we set $f_1(x) = e^{-x^2}$, $f_2(x) = 0$ in the wave equation (5.7):

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t > 0, \quad -\infty < x < \infty, \\ u(x,0) = e^{-x^2}, & -\infty < x < \infty, \\ u_t(x,0) = 0, & -\infty < x < \infty. \end{cases}$$

According to (5.8) or (5.9), we obtain

$$u(x,t) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right]. \tag{5.12}$$

We note that the Gaussian in (5.11) spreads and decays with time while Gaussians in (5.12) propagate preserving their shapes. See Fig. 5.1.

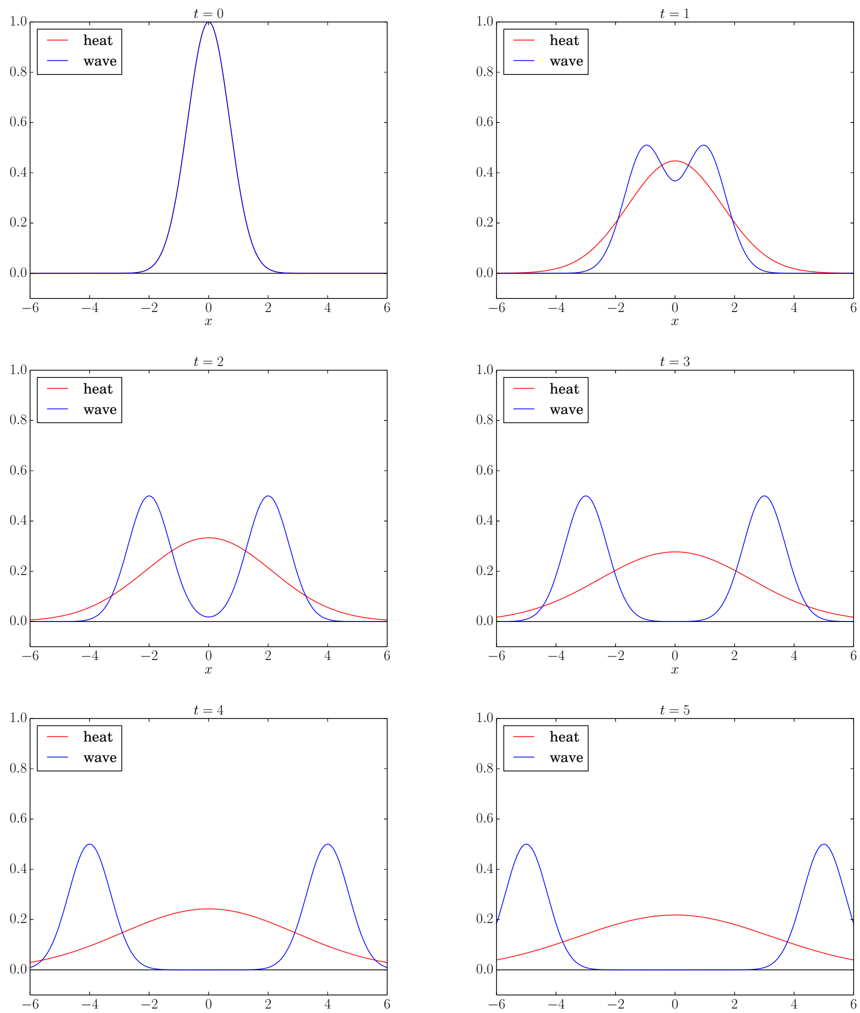

Fig. 5.1 The solution to the heat equation with $K = 1$ (5.11) and the solution to the wave equation with $c = 1$ (5.12) are compared at $t = 0, 1, 2, 3, 4, 5$.