Chapter 4
PDEs in spherical coordinates

Spherically symmetric solutions [1]

Using spherical coordinates, we have

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta, \tag{4.1}$$

where $r \geq 0$, $0 \leq \theta \leq \pi$, and $-\pi \leq \varphi \leq \pi$.

Let us consider the Laplacian. We recall that in cylindrical coordinates (or polar coordinates) we have

$$x = \rho\cos\varphi, \qquad y = \rho\sin\varphi,$$

and

$$u_{xx} + u_{yy} = u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi}.$$

In spherical coordinates we have

$$z = r\cos\theta, \qquad \rho = r\sin\theta.$$

We can then read off

$$u_{zz} + u_{\rho\rho} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Hence we obtain

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi}.$$

We note that

$$r = \sqrt{\rho^2 + z^2} \quad \Rightarrow \quad \frac{\partial r}{\partial \rho} = \frac{\rho}{r},$$

$$\tan\theta = \frac{\rho}{z} \quad \Rightarrow \quad \frac{d\tan\theta}{d\theta}\frac{\partial\theta}{\partial\rho} = \frac{1}{z} \quad \Rightarrow \quad \frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial\rho} = \frac{1}{r\cos\theta} \quad \Rightarrow \quad \frac{\partial\theta}{\partial\rho} = \frac{\cos\theta}{r}.$$

Therefore we obtain

Winter 2014 Math 454 Sec 2
Boundary Value Problems for Partial Differential Equations

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[1] This section corresponds to §4.1 of the textbook.

$$u_{xx}+u_{yy}+u_{zz}=u_{rr}+\frac{2}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+\frac{\cos\theta}{r^2\sin\theta}u_\theta+\frac{1}{r^2\sin^2\theta}u_{\varphi\varphi}.$$

The Laplacian is obtained as

$$\begin{aligned}\Delta u=\nabla^2 u&=u_{xx}+u_{yy}+u_{zz}\\&=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta)+\frac{1}{r^2\sin^2\theta}u_{\varphi\varphi}.\end{aligned}$$

We can also derive the Laplacian directly without using cylindrical coordinates. For u_x, we differentiate (4.1) with respect to x.

$$1=\frac{\partial r}{\partial x}\sin\theta\cos\varphi+\frac{\partial\theta}{\partial x}r\cos\theta\cos\varphi-\frac{\partial\varphi}{\partial x}r\sin\theta\sin\varphi, \tag{4.2}$$

$$0=\frac{\partial r}{\partial x}\sin\theta\sin\varphi+\frac{\partial\theta}{\partial x}r\cos\theta\sin\varphi+\frac{\partial\varphi}{\partial x}r\sin\theta\cos\varphi, \tag{4.3}$$

$$0=\frac{\partial r}{\partial x}\cos\theta-\frac{\partial\theta}{\partial x}r\sin\theta. \tag{4.4}$$

We obtain

$$\begin{aligned}\frac{\partial r}{\partial x}&=\sin\theta\cos\varphi \quad\Leftarrow\quad (4.2)\times\sin\theta\cos\varphi+(4.3)\times\sin\theta\sin\varphi+(4.4)\times\cos\theta,\\ \frac{\partial\theta}{\partial x}&=\frac{1}{r}\cos\theta\cos\varphi \quad\Leftarrow\quad (4.2)\times\cos\theta\cos\varphi+(4.3)\times\cos\theta\sin\varphi-(4.4)\times\sin\theta,\\ \frac{\partial\varphi}{\partial x}&=\frac{-\sin\varphi}{r\sin\theta} \quad\Leftarrow\quad (4.2)\times(-1)\sin\theta\sin\varphi+(4.3)\times\sin\theta\cos\varphi.\end{aligned}$$

Thus we have

$$u_x=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}+\frac{\partial u}{\partial\varphi}\frac{\partial\varphi}{\partial x}=\sin\theta\cos\varphi\frac{\partial u}{\partial r}+\frac{1}{r}\cos\theta\cos\varphi\frac{\partial u}{\partial\theta}-\frac{\sin\varphi}{r\sin\theta}\frac{\partial u}{\partial\varphi},$$

and $u_{xx}=\frac{\partial u_x}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u_x}{\partial\theta}\frac{\partial\theta}{\partial x}+\frac{\partial u_x}{\partial\varphi}\frac{\partial\varphi}{\partial x}$. We can similarly calculate u_{yy} and u_{zz}, and obtain $u_{xx}+u_{yy}+u_{zz}$. However, this requires a lot more lengthy calculations even though actually it is doable.

Example 1. The temperature of the earth can be formulated as

$$\begin{cases} u_t=K\nabla^2 u, & -\infty<t<\infty,\quad 0\le r<a,\\ u(r,\theta,\varphi,t)=e^{i\omega t}, & -\infty<t<\infty,\quad r=a,\end{cases}$$

where a is the radius of the earth. Taking into account the spherical symmetry, we look for the solution in the form $u=u(r,t)$. Then $\nabla^2 u=u_{rr}+\frac{2}{r}u_r$. Define

$$w(r,t) = ru(r,t).$$

The new function satisfies

$$\begin{cases} w_t = Kw_{rr}, & -\infty < t < \infty, \quad 0 \le r < a, \\ w(a,t) = ae^{i\omega t}, & -\infty < t < \infty, \\ w(0,t) = 0, & -\infty < t < \infty. \end{cases}$$

By assuming $w(r,t) = e^{i\omega t}e^{\gamma r}$, we obtain $\gamma = \pm c(1+i)$, where $c = \sqrt{\omega/2K}$. Hence,

$$u(r,t) = \frac{a}{r}e^{i\omega t}\frac{e^{c(1+i)r} - e^{-c(1+i)r}}{e^{c(1+i)a} - e^{-c(1+i)a}}.$$

Legendre polynomials [2]

Let us begin with

$$\Theta''(\theta) + \cot\theta\,\Theta'(\theta) + \mu\Theta(\theta) = 0.$$

By writing $s = \cos\theta$, $y(s) = \Theta(\theta)$, we have the Legendre equation (recall Example 9 in Chapter 2)

$$(1-s^2)y'' - 2sy' + \mu y = \frac{d}{ds}\left[(1-s^2)\frac{dy}{ds}\right] + \mu y = 0.$$

Suppose $y(s) = \sum_{n=0}^{\infty} a_n s^n$ be a solution. Then we obtain

$$a_{n+2} = \frac{n(n+1)-\mu}{(n+2)(n+1)}a_n, \quad n = 0,1,2,\cdots.$$

If μ is of the form $k(k+1)$ $(k = 0,1,2,\cdots)$, then $y(s)$ is a polynomial of degree k. Otherwise the series for y diverges. For given k, we have

$$a_k = \frac{(2k)!}{2^k(k!)^2}, \quad \begin{cases} a_1 = 0, & k \text{ even}, \\ a_0 = 0, & k \text{ odd}. \end{cases}$$

The value of a_k is determined so that $y(1) = 1$. Therefore the solution to

$$(1-s^2)y'' - 2sy' + k(k+1)y = 0$$

is a polynomial of degree k, i.e.,

$$y(s) = P_k(s) = \sum_{n=0}^{k} a_n s^n.$$

[2] This section corresponds to §4.2 of the textbook.

This polynomial $P_k(s)$ is the Legendre polynomial of degree k. We see that $P_k(s)$ is even for $k = 0, 2, 4, \cdots$, and odd for $k = 1, 3, 5, \cdots$.

Since $P_k(s)$ are eigenfunctions of a Sturm-Liouville eigenvalue problem, they are orthogonal to each other:

$$\int_{-1}^{1} P_n(s) P_{n'}(s) ds = \int_0^{\pi} P_n(\cos\theta) P_{n'}(\cos\theta) \sin\theta d\theta = 0, \quad n \neq n'.$$

Let us expand the polynomial $(d/ds)^s (s^2-1)^k$ with Legendre polynomials.

$$\left(\frac{d}{ds}\right)^s (s^2-1)^k = \sum_{j=0}^{k} c_j P_j(s).$$

Using the above orthogonality relations, we obtain $c_j = 0$ for $j < k$ and $c_k = 2^k k!$. Thus, we obtain Rodrigues' formula:

$$P_k(s) = \frac{1}{2^k k!} \left(\frac{d}{ds}\right)^k (s^2-1)^k.$$

Using Rodrigues' formula, we find

$$\int_{-1}^{1} P_k(s)^2 ds = \frac{2}{2k+1}.$$

The Legendre polynomials satisfy the following three-term recurrence relation.

$$nP_n(s) = (2n-1)sP_{n-1}(s) - (n-1)P_{n-2}(s) \quad n = 2, 3, \cdots,$$
$$P_0(s) = 1, \quad P_1(s) = s.$$

We have

$$P_0(s) = 1, \quad P_1(s) = s, \quad P_2(s) = \frac{1}{2}\left(3s^2 - 1\right), \ldots.$$

These Legendre polynomials are plotted in Fig. 4.1.

We consider the Legendre polynomial expansions. Consider the expansion of a function $f(s)$ in a series of Legendre polynomials.

$$f(s) = \sum_{k=0}^{\infty} A_k P_k(s), \quad -1 \leq s \leq 1.$$

Then we have

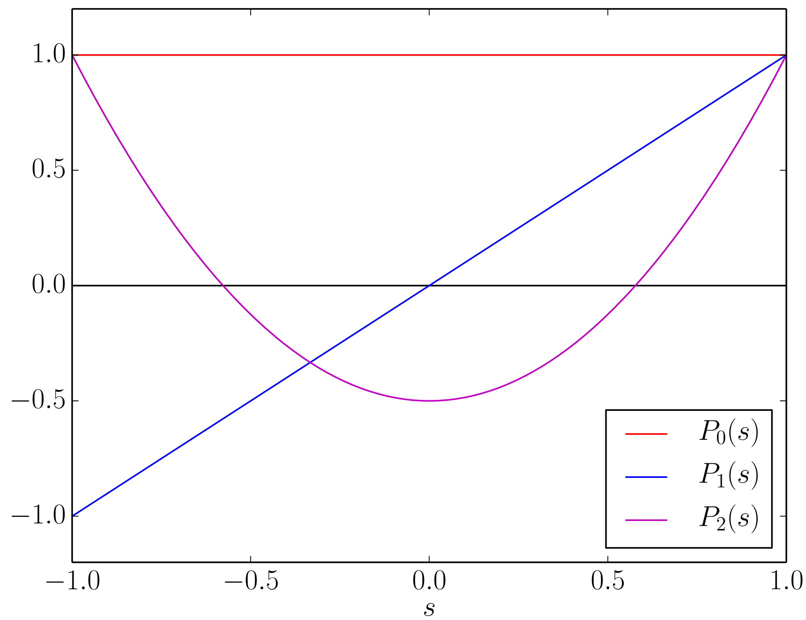

Fig. 4.1 Legendre polynomials $P_0(s)$, $P_1(s)$, and $P_2(s)$ are plotted.

$$\int_{-1}^{1} f(s)P_j(s)ds = \int_{-1}^{1} \sum_{k=0}^{\infty} A_k P_k(s)P_j(s)ds = \frac{2A_j}{2j+1}.$$

Therefore we obtain $A_j = [(2j+1)/2] \int_{-1}^{1} f(s)P_j(s)ds$.

Theorem 1. *Let $f(s)$, $-1 < s < 1$, be a piecewise smooth function. Let*

$$A_k = \frac{2k+1}{2} \int_{-1}^{1} f(s)P_k(s)ds, \quad k = 0, 1, 2, \cdots.$$

Then

$$\sum_{k=0}^{\infty} A_k P_k(s) = \frac{1}{2}\left[f(s+0) + f(s-0)\right], \quad -1 < s < 1.$$

At $s = 1$ ($s = -1$), the series converges to $f(1-0)$ ($f(-1+0)$).

Example 2. Let us find the expansion of the function $f(s) = 1$ in a series of Legendre polynomials. If we write $1 = \sum_{k=0}^{\infty} A_k P_k(s)$, we obtain

$$A_k = \frac{2k+1}{2}\int_{-1}^{1} P_k(s)ds = \frac{2k+1}{2}\int_{-1}^{1} P_0(s)P_k(s)ds = \delta_{k0}.$$

Therefore, $1 = \sum_{k=0}^{\infty} A_k P_k(s) = P_0(s) = 1$.

On the interval $0 < s < 1$, we can define an odd function or even function by extending $f(s)$ and consider the expansion on $(-1,1)$ just like Fourier sine (cosine) series.

Example 3. Consider $f(s) = 1, 0 < s < 1$, in a series of the form $\sum_{n=0}^{\infty} A_{2n+1}P_{2n+1}(s)$. We define $f_O(s) = 1$ $(0 < s < 1)$, -1 $(-1 < s < 0)$, and 0 $(s = 0)$. Then

$$f_O(s) = \sum_{k=0}^{\infty} A_k P_k(s) = \sum_{n=0}^{\infty} A_{2n+1}P_{2n+1}(s).$$

We have

$$A_{2n+1} = \frac{4n+3}{2}\int_{-1}^{1} f_O(s)P_{2n+1}(s)ds = (4n+3)\int_{0}^{1} P_{2n+1}(s)ds = \frac{(4n+3)P_{2n+1}'(0)}{(2n+1)(2n+2)}.$$

where we used the Legendre equation $\frac{d}{ds}\left[(1-s^2)\frac{dP_k(s)}{ds}\right] + k(k+1)P_k(s) = 0$.

Associated Legendre polynomials

More generally we have

$$\Theta''(\theta) + \cot\theta\,\Theta'(\theta) + \left(k(k+1) - \frac{m^2}{\sin^2\theta}\right)\Theta(\theta) = 0.$$

By using $s = \cos\theta$ and $y(s) = \Theta(\theta)$, the above equation becomes

$$(1-s^2)y'' - 2sy' + \left(k(k+1) - \frac{m^2}{1-s^2}\right)y = 0.$$

This is the associated Legendre equation (recall Example 9 in Chapter 2). Therefore $y(s) = P_{k,m}(s)$ or $\Theta(\theta) = P_{k,m}(\cos\theta)$. The associated Legendre polynomial of degree k and order m is obtained as

$$P_{k,m}(s) = (1-s^2)^{m/2}\left(\frac{d}{ds}\right)^m P_k(s),$$

where $k = 0,1,2,\cdots$, $m = 0,1,\cdots,k$, and $s \in [-1,1]$. We have

$$P_{1,0}(s) = P_1(s), \quad P_{1,1}(s) = \sqrt{1-s^2}, \quad P_{2,0}(s) = P_2(s), \quad P_{2,1}(s) = 3s\sqrt{1-s^2}, \quad P_{2,2}(s) = 3(1-s^2).$$

These associated Legendre polynomials are shown in Fig. 4.2. We also note that

$$\int_{-1}^{1} P_{n,m}(s) P_{n',m}(s) ds = 0, \quad n \neq n',$$
$$\int_{-1}^{1} P_{n,m}(s)^2 ds = \frac{(n+m)!}{(n-m)!} \frac{2}{2k+1}.$$

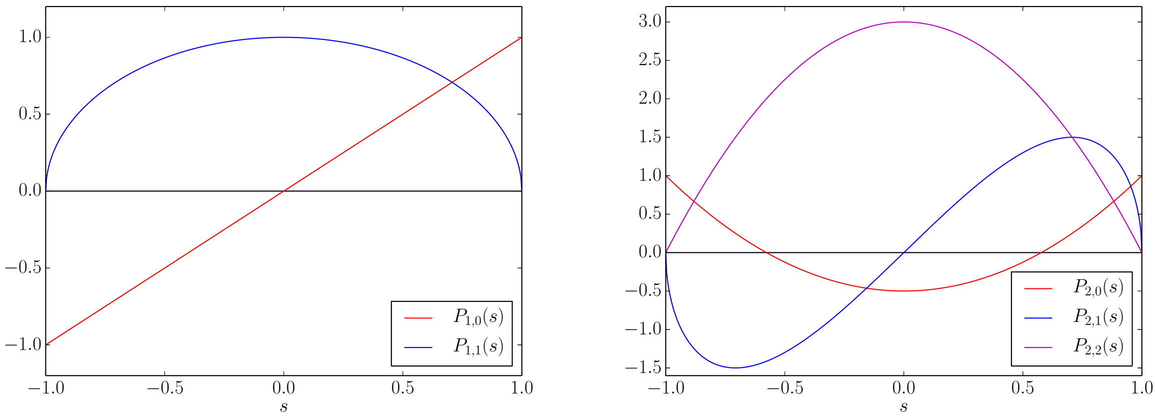

Fig. 4.2 (Left) associated Legendre polynomials $P_{1,0}(s)$, $P_{1,1}(s)$. (Right) associated Legendre polynomials $P_{2,0}(s)$, $P_{2,1}(s)$, and $P_{2,2}(s)$ are plotted.

Laplace's equation in spherical coordinates [3]

Let us solve Laplace's equation with axial symmetry.

Example 4. Let us solve the following problem.

$$\begin{cases} \nabla^2 u = 0, & 0 \leq r < a, \quad 0 \leq \theta \leq \pi, \quad -\pi \leq \varphi \leq \pi, \\ u = G(\theta), & r = a, \quad 0 \leq \theta \leq \pi, \quad -\pi \leq \varphi \leq \pi, \end{cases}$$

where

$$G(\theta) = \begin{cases} 1, & \text{if} \quad 0 < \theta < \dfrac{\pi}{2}, \\ 0, & \text{if} \quad \dfrac{\pi}{2} < \theta < \pi. \end{cases}$$

Note that the solution u must be independent of φ. In spherical coordinates, the Laplacian is written as

[3] This section corresponds to §4.3 of the textbook.

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta u_\theta) + \frac{1}{r^2 \sin^2\theta} u_{\varphi\varphi}$$
$$= u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}(u_{\theta\theta} + \cot\theta u_\theta).$$

We use separation of variables: $u(r,\theta) = R(r)\Theta(\theta)$. By introducing the separation constant λ, we get

$$\Theta'' + \cot\theta\Theta' + \lambda\Theta = 0, \tag{4.5}$$
$$R'' + \frac{2}{r}R' - \frac{\lambda}{r^2}R = 0. \tag{4.6}$$

If λ is not of the form $\lambda = k(k+1)$ $(k = 0, 1, 2, \cdots)$, (4.5) doesn't have bounded solutions (cf. the previous section). So, we put

$$\lambda = k(k+1).$$

Then we obtain

$$\Theta(\theta) = P_k(\cos\theta), \quad (1-s^2)P_k''(s) - 2sP_k'(s) + k(k+1)P_k(s) = 0.$$

By setting $R = r^\gamma$ in (4.6), we obtain $\gamma = k, -(k+1)$. The general solution for $R(r)$ is written as

$$R(r) = Ar^k + \frac{B}{r^{k+1}}.$$

To have bounded solutions, we choose $B = 0$. Hence the general solution is then written as

$$u(r,\theta) = \sum_{k=0}^{\infty} A_k r^k P_k(\cos\theta).$$

Using the orthogonality relations of Legendre polynomials, we have

$$A_k = \frac{(2k+1)/2}{a^k}\int_0^\pi G(\theta)P_k(\cos\theta)\sin\theta\, d\theta = \frac{k+\frac{1}{2}}{a^k}\int_0^1 P_k(s)ds.$$

Noting $[(1-s^2)P_k'(s)]' + k(k+1)P_k(s) = 0$, the integral on the right-hand side is calculated as

$$\int_0^1 P_k(s)ds = \frac{-1}{k(k+1)}\int_0^1 [(1-s^2)P_k'(s)]'ds = \frac{-1}{k(k+1)}\ (1-s^2)P_k'(s)\big|_0^1 = \frac{1}{k(k+1)}P_k'(0).$$

We obtain $A_0 = \frac{1}{2}$ and $A_k = \frac{k+\frac{1}{2}}{k(k+1)}\frac{P_k'(0)}{a^k}$ $(k = 1, 2, \cdots)$. Finally we obtain

$$u(r,\theta) = \frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{r}{a}\right)^k P_k'(0)\frac{k+\frac{1}{2}}{k(k+1)}P_k(\cos\theta).$$