Chapter 3
PDEs in cylindrical coordinates

Laplace's equation and applications [1]

In cylindrical coordinates we use ρ, φ, z instead of x, y, z:

$$x = \rho \cos\varphi, \quad y = \rho \sin\varphi, \quad z = z.$$

In rectangular coordinates the Laplacian was given by $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. Here we will compute the Laplacian in cylindrical coordinates.

For a function $u(\rho, \varphi, z)$ we have

$$\begin{cases} u_x = \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \rho}\dfrac{\partial \rho}{\partial x} + \dfrac{\partial u}{\partial \varphi}\dfrac{\partial \varphi}{\partial x} = \cos\varphi \dfrac{\partial u}{\partial \rho} - \dfrac{\sin\varphi}{\rho}\dfrac{\partial u}{\partial \varphi}, \\ u_y = \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial \rho}\dfrac{\partial \rho}{\partial y} + \dfrac{\partial u}{\partial \varphi}\dfrac{\partial \varphi}{\partial y} = \sin\varphi \dfrac{\partial u}{\partial \rho} + \dfrac{\cos\varphi}{\rho}\dfrac{\partial u}{\partial \varphi}, \end{cases}$$

where we used

$$\rho^2 = x^2 + y^2 \quad \Rightarrow \quad 2\rho\frac{\partial \rho}{\partial x} = 2x, \quad 2\rho\frac{\partial \rho}{\partial y} = 2y \quad \Rightarrow \quad \frac{\partial \rho}{\partial x} = \frac{x}{\rho} = \cos\varphi, \quad \frac{\partial \rho}{\partial y} = \frac{y}{\rho} = \sin\varphi,$$

$$y = \rho\sin\varphi \quad \Rightarrow \quad 0 = \frac{\partial \rho}{\partial x}\sin\varphi + \rho\cos\varphi\frac{\partial \varphi}{\partial x} = \cos\varphi\sin\varphi + \rho\cos\varphi\frac{\partial \varphi}{\partial x} \quad \Rightarrow \quad \frac{\partial \varphi}{\partial x} = -\frac{\sin\varphi}{\rho},$$

and

$$x = \rho\cos\varphi \quad \Rightarrow \quad 0 = \frac{\partial \rho}{\partial y}\cos\varphi - \rho\sin\varphi\frac{\partial \varphi}{\partial y} = \sin\varphi\cos\varphi - \rho\sin\varphi\frac{\partial \varphi}{\partial y} \quad \Rightarrow \quad \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{\rho}.$$

Thus the second derivatives are obtained as

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \cos^2\varphi\dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{2\cos\varphi\sin\varphi}{\rho^2}\dfrac{\partial u}{\partial \varphi} - \dfrac{2\sin\varphi\cos\varphi}{\rho}\dfrac{\partial^2 u}{\partial \rho\partial \varphi} + \dfrac{\sin^2\varphi}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{\sin^2\varphi}{\rho^2}\dfrac{\partial^2 u}{\partial \varphi^2}, \\ \dfrac{\partial^2 u}{\partial y^2} = \sin^2\varphi\dfrac{\partial^2 u}{\partial \rho^2} - \dfrac{2\sin\varphi\cos\varphi}{\rho^2}\dfrac{\partial u}{\partial \varphi} + \dfrac{2\sin\varphi\cos\varphi}{\rho}\dfrac{\partial^2 u}{\partial \rho\partial \varphi} + \dfrac{\cos^2\varphi}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{\cos^2\varphi}{\rho^2}\dfrac{\partial^2 u}{\partial \varphi^2}. \end{cases}$$

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Boundary Value Problems for Partial Differential Equations

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[1] This section corresponds to §3.1 of the textbook.

Hence,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2}.$$

The Laplacian is obtained as

$$\Delta u=\nabla^2 u=\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2}+\frac{\partial^2 u}{\partial z^2}.$$

Example 1. Let us find separated solutions of Laplace's equation $\nabla^2 u=0$ in cylindrical coordinates, defined for $\rho>0$, $-\pi\le\varphi\le\pi$. Assume that u is smooth and is independent of z.

By plugging $u(\rho,\varphi)=R(\rho)\Phi(\varphi)$ into $\nabla^2 u=0$, we obtain

$$0=u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\varphi\varphi}=R''\Phi+\frac{1}{\rho}R'\Phi+\frac{1}{\rho^2}R\Phi''.$$

Dividing by $R\Phi$ and multiplying by ρ^2, we have

$$0=\rho^2\frac{R''+(1/\rho)R'}{R}+\frac{\Phi''}{\Phi}.$$

By introducing the separation constant λ, we have

$$\begin{cases}\Phi''+\lambda\Phi=0, \quad \Phi(-\pi)=\Phi(\pi), \quad \Phi'(-\pi)=\Phi'(\pi),\\ R''+\dfrac{1}{\rho}R'-\dfrac{\lambda}{\rho^2}R=0.\end{cases}$$

Note that λ and Φ are an eigenvalue and an eigenfunction of the Sturm-Liouville problem: $\lambda=m^2$ and $\Phi(\varphi)=A_m\cos m\varphi+B_m\sin m\varphi$ $(m=0,1,2,\dots)$. Separated solutions are obtained as

$$u(\rho,\varphi)=\begin{cases}\rho^m\left(A_m\cos m\varphi+B_m\sin m\varphi\right), & m=1,2,\dots,\\ A_0+B_0\ln\rho, & m=0,\\ \rho^{-m}\left(C_m\cos m\varphi+D_m\sin m\varphi\right), & m=1,2,\dots.\end{cases}$$

In the last two cases we have $|u|\to\infty$ as $\rho\to 0$ and u is not smooth. Therefore,

$$u(\rho,\varphi)=\rho^m\left(A_m\cos m\varphi+B_m\sin m\varphi\right), \quad m=1,2,\dots.$$

Bessel functions [2]

Let us begin with

$$R''(\rho)+\frac{1}{\rho}R'(\rho)+\left(\lambda-\frac{m^2}{\rho^2}\right)R(\rho)=0. \tag{3.1}$$

Let $x=\rho\sqrt{\lambda}$ and $y(x)=R(\rho)$. Then, (3.1) becomes

$$y''+\frac{1}{x}y'+\left(1-\frac{m^2}{x^2}\right)y=0. \tag{3.2}$$

Equation (3.2) is Bessel's equation (recall Example 8 in Chapter 2). Therefore $y(x)=J_m(x)$ or

$$R(\rho)=J_m(\rho\sqrt{\lambda}).$$

Definition 1.

$$J_m(x)=\frac{1}{2\pi i^m}\int_{-\pi}^{\pi}e^{ix\cos\theta}e^{-im\theta}d\theta,\quad m=0,1,2,\ldots. \tag{3.3}$$

Bessel functions $J_m(x)$ behave as shown in Fig. 3.1. We will show in the end of this section that $J_m(x)$ in (3.3) satisfies (3.2).

The following recurrence formula holds.

$$J_m(x)=\frac{x}{2m}\left[J_{m-1}(x)+J_{m+1}(x)\right],\quad m=1,2,\ldots. \tag{3.4}$$

Derivatives are given as (the differentiation formula)

$$J'_m(x)=\frac{1}{2}\left[J_{m-1}(x)-J_{m+1}(x)\right],\quad m=0,1,2,\ldots. \tag{3.5}$$

By considering [(3.5)+(m/x)(3.4)]x^m and [(3.5)−(m/x)(3.4)]x^{-m}, we have

$$\frac{d}{dx}\left[x^mJ_m(x)\right]=x^mJ_{m-1}(x),\quad m=1,2,\cdots, \tag{3.6}$$

$$\frac{d}{dx}\left[x^{-m}J_m(x)\right]=-x^{-m}J_{m+1}(x),\quad m=0,1,2,\cdots. \tag{3.7}$$

Let $\{x_n\}$ be the nonnegative solutions of the equation

$$J_m(x_n)\cos\beta+x_nJ'_m(x_n)\sin\beta=0, \tag{3.8}$$

where $m\geq 0$ and $0\leq\beta\leq\pi/2$.

[2] This section corresponds to §3.2 of the textbook.

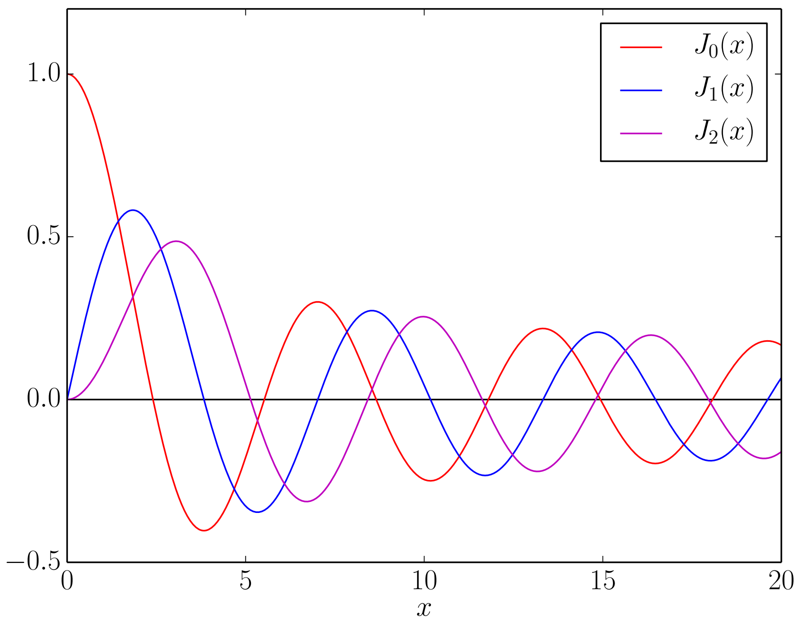

Fig. 3.1 Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$ are plotted.

Then we have the following orthogonality relations.

$$\begin{cases} \int_0^1 J_m(xx_{n_1})J_m(xx_{n_2})xdx = 0, & n_1 \neq n_2, \\ \int_0^1 J_m(xx_n)^2 xdx = \frac{1}{2}J_{m+1}(x_n)^2, & \text{if} \quad \beta = 0, \\ \int_0^1 J_m(xx_n)^2 xdx = \frac{x_n^2 - m^2 + \cot^2\beta}{2x_n^2}J_m(x_n)^2, & \text{if} \quad 0 < \beta \leq \frac{\pi}{2}. \end{cases} \tag{3.9}$$

From the Sturm-Liouville theory, the first equation (orthogonality) holds. For the second and third equations, we multiply (3.1) by $2\rho^2 R'$ and set $\lambda = x_n^2$.

$$2\rho^2 R'R'' + 2\rho(R')^2 + (x_n^2\rho^2 - m^2)2RR' = 0.$$

We can rewrite this as

$$\left[(\rho R')^2\right]' + (x_n^2\rho^2 - m^2)(R^2)' = 0.$$

By integrating both sides and using integration by parts, we get

$$(\rho R')^2\big|_{\rho=1} - (\rho R')^2\big|_{\rho=0} + (x_n^2\rho^2 - m^2)R^2\big|_0^1 - \int_0^1 2x_n^2\rho R^2 d\rho = 0.$$

Note that $R(\rho) = J_m(\rho x_n)$ and $R'(\rho) = x_n J_m'(x)|_{x=\rho x_n}$. Hence $R(0) = J_m(0) = 0$ $(m = 1, 2, \cdots)$. We obtain

$$\left[x_n J_m'(x_n)\right]^2 + (x_n^2 - m^2)J_m(x_n)^2 - 2x_n^2 \int_0^1 J_m(\rho x_n)^2 \rho d\rho = 0.$$

Therefore, when $\beta = 0$ or $J_m(x_n) = 0$, we have

$$\int_0^1 J_m(\rho x_n)^2 \rho d\rho = \frac{x_n^2 \left[J_m'(x_n)\right]^2}{2x_n^2} = \frac{\left[\frac{m}{x_n} J_m(x_n) - J_{m+1}(x_n)\right]^2}{2} = \frac{J_{m+1}(x_n)^2}{2},$$

and when $0 < \beta \le \pi/2$ or $J_m(x_n)\cot\beta + x_n J_m'(x_n) = 0$, we have

$$\begin{aligned}\int_0^1 J_m(\rho x_n)^2 \rho d\rho &= \frac{x_n^2 \left[\frac{-1}{x_n} J_m(x_n)\cot\beta\right]^2 + (x_n^2 - m^2)J_m(x_n)^2}{2x_n^2} \\ &= \frac{(x_n^2 - m^2 + \cot^2\beta)J_m(x_n)^2}{2x_n^2}.\end{aligned}$$

Let us consider the expansion of a piecewise smooth function $f(x)$, $0 < x < 1$, in a series of the form

$$f(x) = \sum_{n=1}^{\infty} A_n J_m(x x_n), \quad 0 < x < 1, \tag{3.10}$$

where $\{x_n\}$ are the nonnegative solutions of $J_m(x)\cos\beta + xJ_m'(x)\sin\beta = 0$. This is called a Fourier-Bessel expansion. By multiplying (3.10) by $J_m(xx_n)$ and integrating both sides, we obtain

$$A_n = \frac{\int_0^1 f(x) J_m(x x_n) x dx}{\int_0^1 J_m(x x_n)^2 x dx}, \quad n = 1, 2, \cdots. \tag{3.11}$$

Theorem 1. *Let $m \ge 0$, $0 \le \beta \le \pi/2$, and let $\{x_n : n \ge 1\}$ be the nonnegative solutions of (3.8). If $f(x)$, $0 < x < 1$, is a piecewise smooth function, define $\{A_n : n \ge 1\}$ by (3.11). Then the series $\sum_{n=1}^{\infty} A_n J_m(xx_n)$ converges for each $x \in [0,1]$, and the sum is $\frac{1}{2}\left[f(x+0) + f(x-0)\right]$ for $0 < x < 1$.*

Example 2. Let us compute the Fourier-Bessel expansion of the function $f(x) = 1$, $0 < x < 1$, where $m = 0$ and $\beta = 0$. We have $1 = \sum_{n=1}^{\infty} A_n J_0(xx_n)$, where $J_0(x_n) = 0$ and

$$\int_0^1 xJ_0(xx_n)dx = A_n \int_0^1 J_0(xx_n)^2 xdx, \quad n = 1, 2, \cdots.$$

Using (3.6) and (3.9), we obtain

$$A_n = \frac{\frac{1}{x_n^2}\int_0^{x_n} tJ_0(t)dt}{\int_0^1 J_0(xx_n)^2 xdx} = \frac{\frac{1}{x_n^2} tJ_1(t)\Big|_0^{x_n}}{\int_0^1 J_0(xx_n)^2 xdx} = \frac{\frac{1}{x_n} J_1(x_n)}{\frac{1}{2} J_1(x_n)^2} = \frac{2}{x_n J_1(x_n)}.$$

Therefore,

$$1 = 2\sum_{n=1}^{\infty} \frac{J_0(xx_n)}{x_n J_1(x_n)}.$$

Finally we show that Bessel functions (3.3) are solutions to Bessel's equation. Let $y = \sum_{n=0}^{\infty} a_n x^{n+\gamma}$ ($a_0 \neq 0$, $\gamma \geq 0$) be a solution to (3.2). We obtain

$$\left(\gamma^2 - m^2\right) a_0 x^{\gamma} + \left((1+\gamma)^2 - m^2\right) a_1 x^{\gamma+1} + \sum_{n=2}^{\infty} \left[\left((n+\gamma)^2 - m^2\right) a_n + a_{n-2}\right] x^{n+\gamma} = 0.$$

Hence,

$$\gamma = m, \quad a_1 = 0, \quad a_n = \frac{-a_{n-2}}{n(n+2m)} \ (n \geq 2).$$

We obtain

$$y = a_0 x^m \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2(2+2m)4(4+2m)\cdots 2n(2n+2m)}\right].$$

Let us choose $a_0 = 1/m!2^m$. Then,

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+m}}{2^{m+2n}(m+n)!n!}. \tag{3.12}$$

We rewrite (3.3) using $e^{ix\cos\theta} = \sum_{n=0}^{\infty} (ix\cos\theta)^n/n!$ and introducing j as $n = m + 2j$.

$$J_m(x) = \frac{1}{2\pi i^m} \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \int_{-\pi}^{\pi} \cos^n \theta e^{-im\theta} d\theta.$$

The integral is nonzero only for $n = m, m+2, m+4, \ldots$. We introduce j ($n = m + 2j$).

$$J_m(x) = \frac{1}{2\pi i^m} \sum_{j=0}^{\infty} \frac{(ix)^{m+2j}}{(m+2j)!} \int_{-\pi}^{\pi} \cos^{m+2j} \theta e^{-im\theta} d\theta$$

$$= \frac{1}{i^m} \sum_{j=0}^{\infty} \frac{(ix)^{m+2j}}{(m+2j)!} \frac{1}{2^{m+2j}} \begin{pmatrix} m+2j \\ j \end{pmatrix} = \frac{1}{i^m} \sum_{j=0}^{\infty} \frac{(ix)^{m+2j}}{2^{m+2j}(m+j)!j!} = (3.12).$$

The vibrating drumhead [3]

Let us consider small transverse vibrations of a circular membrane.

$$\begin{cases} u_{tt} = c^2\nabla^2 u = c^2\left(u_{\rho\rho} + \dfrac{1}{\rho}u_\rho + \dfrac{1}{\rho^2}u_{\varphi\varphi}\right), & 0 \le \rho < a, \quad t > 0, \\ u(\rho,\varphi,t) = 0, & \rho = a, \quad t > 0, \\ u(\rho,\varphi,0) = 1, \quad u_t(\rho,\varphi,0) = 0, & 0 \le \rho < a. \end{cases}$$

where $c^2 = T_0/\rho$ (cf. Chapter 2).

We look for separated solutions in the form

$$u(\rho,\varphi,t) = R(\rho)\Phi(\varphi)T(t).$$

By introducing separation constants as $-\lambda = (1/c^2)T''/T$ and $-\mu = \Phi''/\Phi$, we obtain

$$\Phi''(\varphi) + \mu\Phi(\varphi) = 0, \quad \Phi(-\pi) = \Phi(\pi), \quad \Phi'(-\pi) = \Phi'(\pi), \tag{3.13}$$

$$R''(\rho) + \frac{1}{\rho}R'(\rho) + \left(\lambda - \frac{\mu}{\rho^2}\right)R(\rho) = 0, \quad R(a) = 0, \tag{3.14}$$

$$T''(t) + \lambda c^2 T(t) = 0. \tag{3.15}$$

In (3.13), nontrivial solutions are obtained when (i) $\mu > 0$, $\sqrt{\mu} = 1,2,\cdots$, and (ii) $\mu = 0$:

$$\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi, \quad m = 0,1,2,\cdots.$$

With $\mu = m^2$ $(m = 0,1,2,\cdots)$, we obtain $R(\rho) = J_m(\rho\sqrt{\lambda})$. For $R(a) = 0$, we obtain $\sqrt{\lambda} = x_n^{(m)}/a$ where $x_n^{(m)} > 0$, $J_m(x_n^{(m)}) = 0$. The separated solutions are obtained as

$$u(\rho,\varphi,t) = J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A\cos m\varphi + B\sin m\varphi)\left(\tilde{A}\cos\frac{ctx_n^{(m)}}{a} + \tilde{B}\sin\frac{ctx_n^{(m)}}{a}\right). \tag{3.16}$$

We will now take the initial conditions into account. The general solution is given as a linear combination (superposition) of (3.16). To satisfy $u_t(\rho,\varphi,0) = 0$, we set $\tilde{B} = 0$. Now the solution is written as

$$u(\rho,\varphi,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\{\left[A_{mn}J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\right]\cos m\varphi + \left[B_{mn}J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\right]\sin m\varphi\right\}\cos\frac{ctx_n^{(m)}}{a}.$$

Consider the Fourier series $u(\rho,\varphi,0) = 1 = A_0 + \sum_{m=1}^{\infty} A_m\cos(mx) + B_m\sin(mx)$. We can readily find $A_0 = 1, \quad A_m = B_m = 0 \quad (m \ge 1)$. (Of course we can also obtain them as $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1dx$, $A_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(mx)dx$, and $B_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(mx)dx$.) Therefore,

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[3] This section corresponds to §3.3 of the textbook.

$$A_{mn} = B_{mn} = 0, \qquad \text{for} \quad m \geq 1,$$

and

$$u(\rho,\varphi,t) = \sum_{n=1}^{\infty} A_{0n} J_0\left(\frac{\rho x_n^{(0)}}{a}\right) \cos\frac{ctx_n^{(0)}}{a}.$$

In Example 2, we calculated the Fourier-Bessel expansion $1 = 2\sum_{n=1}^{\infty} \frac{J_0(xx_n^{(0)})}{x_n^{(0)} J_1(x_n^{(0)})}$. By comparison, finally we obtain

$$u(\rho,\varphi,t) = 2\sum_{n=1}^{\infty} \frac{1}{x_n^{(0)} J_1(x_n^{(0)})} J_0\left(\frac{\rho x_n^{(0)}}{a}\right) \cos\frac{ctx_n^{(0)}}{a}.$$

Heat flow in the infinite cylinder [4]

Let us consider the heat transfer in the infinite cylinder $0 \leq \rho < \rho_{\max}$. We will solve the heat equation in polar coordinates

$$\begin{cases} u_t = K\nabla^2 u, & t > 0, \quad 0 \leq \rho < \rho_{\max}, \quad -\pi \leq \varphi \leq \pi, \\ u(\rho_{\max},\varphi,t) = T_1, & t > 0, \quad -\pi \leq \varphi \leq \pi, \\ u(\rho,\varphi,0) = T_2, & 0 \leq \rho < \rho_{\max}, \quad -\pi \leq \varphi \leq \pi, \end{cases}$$

where T_1 and T_2 are positive constants.

Step 1

To find the steady-state solution, we try $U(\rho)$ because the b. c. is independent of φ.

$$K\nabla^2 U = K\left(\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2}\right) = K\left(U'' + \frac{1}{\rho}U'\right) = 0.$$

The general solution is obtained as

$$U(\rho) = A + B\ln\rho.$$

Let us exclude the second term and set $B = 0$ (otherwise $U(0)$ diverges and the initial condition in Step 2 will also diverge). To satisfy $U(\rho_{\max}) = T_1$, we choose $A = T_1$. We thus obtain

$$U(\rho) = T_1.$$

[4] This section corresponds to §3.4 of the textbook.

Step 2

Define $v(\rho,\varphi,t) = u(\rho,\varphi,t) - U(\rho)$. We have

$$\begin{cases} v_t = K\nabla^2 v, & t > 0, \quad 0 \le \rho < \rho_{\max}, \quad -\pi \le \varphi \le \pi, \\ v(\rho_{\max},\varphi,t) = 0, & t > 0, \quad -\pi \le \varphi \le \pi, \\ v(\rho,\varphi,0) = T_2 - U(\rho), & 0 \le \rho < \rho_{\max}, \quad -\pi \le \varphi \le \pi, \end{cases}$$

Step 3

Using separation of variables with $u(\rho,\varphi,t) = R(\rho)\Phi(\varphi)T(t)$, we obtain

$$u(\rho,\varphi,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\rho x_n^{(m)}}{\rho_{\max}}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)\exp\left[-\frac{\left(x_n^{(m)}\right)^2 Kt}{\rho_{\max}^2}\right].$$

Noting that $1 = 2\sum_{n=1}^{\infty}\frac{J_0(xx_n)}{x_n J_1(x_n)}$ $(0 < x < 1,\ J_0(x_n) = 0)$, the solution is obtained as

$$u(\rho,\varphi,t) = T_1 + \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{\rho_{\max}}\right)\exp\left[-\frac{x_n^2 Kt}{\rho_{\max}^2}\right],$$

where $A_n = \frac{2(T_2 - T_1)}{x_n J_1(x_n)}$, $J_0(x_n) = 0$, $n = 1, 2, \cdots$.