Problem Set 11 (4/9, 4/11, 4/14, 4/16, 4/18)
Due on Fri, Apr 18

1) Solve $u_t + u_x = 0$, $-\infty < x < \infty$, $0 < t$ with the initial condition $u(x,0) = \cos x$, $-\infty < x < \infty$.
Solution: $u(x,t) = \cos(x-t)$.

2) Solve $tu_t + xu_x + 2u = 0$, $-\infty < x < \infty$, $1 < t$ with the initial condition $u(x,1) = \sin x$, $-\infty < x < \infty$. (*Hint:* $s = 0$ corresponds to the initial condition.)
Solution: $u(x,t) = \frac{1}{t^2}\sin(x/t)$.

3) Solve $u_{tt} = u_{xx}$, $-\infty < x < \infty$ with the initial conditions $u(x,0) = 0$, $u_t(x,0) = x$ by finding characteristics.
Solution: $u(x,t) = xt$.

4) Find the solution of the heat equation $u_t - Ku_{xx} = h$ for $0 < x < \infty$ satisfying the boundary condition $u_x(0,t) = 0$ and the initial condition $u(x,0) = 0$.
Solution: $u(x,t) = \int_0^t \int_0^\infty [4\pi K(t-s)]^{-1/2} \left[e^{-(x-x')^2/4K(t-s)} + e^{-(x+x')^2/4K(t-s)}\right] h(x',s)dx'ds$.

5) Find the solution of the heat equation $u_t - Ku_{xx} = h$ for $0 < x < L$ satisfying the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$, and the initial condition $u(x,0) = 0$.
Solution: $u(x,t) = \sum_{m=-\infty}^{\infty} \int_0^t \int_0^L [4\pi K(t-s)]^{-1/2}$
$\times \left[e^{-(x-x'-2mL)^2/4K(t-s)} - e^{-(x+x'-(2m+2)L)^2/4K(t-s)}\right] h(x',s)dx'ds$.

6) Find the solution of the heat equation $u_t = Ku_{xx}$, $0 < x < L$, satisfying the boundary conditions $u_x(0,t) = 0$, $u_x(L,t) = 0$ and the initial condition $u(x,0) = f(x)$, a piecewise smooth function.
Solution: $u(x,t) = \frac{1}{\sqrt{4\pi Kt}} \sum_{m=-\infty}^{\infty} \int_0^L \left[e^{-(x-x'-2mL)^2/4Kt} + e^{-(x+x'-(2m+2)L)^2/4Kt}\right] f(x')dx'$.

7) Find the Green's function $G(x,x')$ for $y'' = -f(x)$, $y(0) = 0$, $y'(L) = 0$, such that $y(x) = \int_0^L G(x,x')f(x')dx'$.
Solution: $G(x,x') = \min(x,x')$.

8) Consider $G(x,x')$ which obeys $G_{xx} = -\delta(x-x')$, $G(0,x') = 0$, $G_x(L,x') = 0$. Find $G(x,x')$ in terms of the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ of the Sturm-Liouville problem, $\phi_n'' + \lambda_n\phi_n = 0$, $\phi(0) = 0$, $\phi'(L) = 0$. Normalize $\phi_n(x)$ so that $\int_0^L \phi_n(x)^2dx = 1$.
Solution: This semester we studied a lot and our understanding of PDEs has been significantly deepened. So, I hope you will find the solution to this very last problem by yourself.