**Problem Set 10** (3/28, 3/31, 4/2, 4/4, 4/7)
**Due on Fri, Apr 4**

1) Find the Fourier transform of $f(x)$, where $f(x) = 1$ for $-2 < x < 2$ and $f(x) = 0$ otherwise.
**Solution:** $\tilde{f}(\mu) = \frac{\sin 2\mu}{\pi\mu}$.
2) Find the Fourier transform of $f(x)$, where $f(x) = e^{-x^2/2}$.
**Solution:** $\tilde{f}(\mu) = \frac{1}{\sqrt{2\pi}} e^{-\mu^2/2}$.
3) Find the Fourier transform of $f(x)$, where $f(x) = e^{-(x-2)^2/2}$.
**Solution:** $\tilde{f}(\mu) = \frac{1}{\sqrt{2\pi}} e^{-2i\mu} e^{-\mu^2/2}$.
4) Find the Fourier transform of $f(x) = 1/[1 + (x-3)^2]$.
**Solution:** $\tilde{f}(\mu) = \frac{1}{2} e^{-3i\mu} e^{-|\mu|}$.
5) Consider the following initial-value problem for a diffusion equation with the absorption coefficient $a$ ($> 0$):

$$\begin{cases} u_t = K u_{xx} - au & t > 0,\ -\infty < x < \infty, \\ u = f(x) = e^{-x^2} & t = 0,\ -\infty < x < \infty. \end{cases}$$

Find $u(x,t)$ using the Fourier transform. (*Hint:* You can directly use the Fourier transform. Also you can use the transformation $u(x,t) = e^{-at} w(x,t)$ then use the Fourier transform. Either method is fine.)
**Solution:** $u(x,t) = \frac{1}{\sqrt{4Kt+1}} \exp\left[-\frac{x^2}{4Kt+1}\right] e^{-at}$.
6) Consider the initial-value problem

$$\begin{cases} u_t = K u_{xx} & t > 0,\ x > 0, \\ u_x(0,t) = 0 & t > 0, \\ u(x,0) = f(x) = \begin{cases} 1 & 0 \le x \le L_1, \\ 0 & x > L_1. \end{cases} \end{cases}$$

Find $u(x,t)$ using the method of images. (*Hint:* Define $f_E(x) = f(x)$ $(x \ge 0)$, $f(-x)$ $(x \le 0)$. Then derive $u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi K t}} e^{-(x-y)^2/4Kt} f_E(y) dy$.)
**Solution:** $u(x,t) = \frac{1}{\sqrt{4\pi K t}} \int_0^{L_1} \left\{ \exp\left[-\frac{(x-y)^2}{4Kt}\right] + \exp\left[-\frac{(x+y)^2}{4Kt}\right] \right\} dy$.
7) Consider the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} & t > 0,\ -\infty < x < \infty, \\ u = f_1(x) & t = 0,\ -\infty < x < \infty, \\ u_t = f_2(x) & t = 0,\ -\infty < x < \infty. \end{cases}$$

Derive d'Alembert's formula.
**Solution:** $u(x,t) = \frac{1}{2}\left[f_1(x+ct) + f_1(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(y) dy$.

8) Use d'Alembert's formula to solve the wave equation $u_{tt} = c^2 u_{xx}$ with initial conditions $u(x,0) = 0$ and $u_t(x,0) = 4\cos 5x$.
**Solution:** $u(x,t) = (4/5c)\cos 5x \sin 5ct$.

9) Find the solution of the wave equation $u_{tt} = c^2 u_{xx}$ for $t > 0$ and $x > 0$ satisfying the boundary conditions $u(0,t) = 0$ and the initial conditions $u(x,0) = 0$ and $u_t(x,0) = g(x)$.
**Solution:** Since $u(x,t) = 0$, we extend the function $g(x)$ as

$$g_O(x) = \begin{cases} g(x) & x > 0, \\ 0 & x = 0, \\ -g(-x) & x < 0. \end{cases}$$

Then we use d'Alemberts' formula for

$$\begin{cases} u_{tt} = c^2 u_{xx} & t > 0,\ -\infty < x < \infty, \\ u(x,0) = 0 & -\infty < x < \infty, \\ u_t(x,0) = g_O(x) & -\infty < x < \infty. \end{cases}$$

We obtain $u(x,t) = \frac{1}{2c}\int_{ct-x}^{ct+x} g(y)dy$ for $0 < x < ct$ and $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$ for $x > ct$.