**Winter 2014** Math 454

**Problem Set 8** (3/12, 3/14, 3/17)
**Due on Fri, Mar 28**

1) Let $\{x_n\}$ be the nonnegative solutions to $J_m(x_n)\cos\beta + x_n J_m'(x_n)\sin\beta = 0$, where $m \geq 0$ and $0 \leq \beta \leq \pi/2$. Prove

$$\int_0^1 J_m(xx_{n_1})J_m(xx_{n_2})x\,dx = 0 \quad n_1 \neq n_2.$$

**Solution:** If we define $y_i(x) = J_m(xx_{n_i})$, then the Bessel equation becomes $(xy_i')' + \left(xx_{n_i}^2 - \frac{m^2}{x}\right)y_i = 0$. We then multiply the equation for $y_1$ by $y_2$ and integrate both sides over $x$. Interchanging the roles of $y_1$ and $y_2$ and subtracting the resulting equations leaves

$$(y_1'y_2 - y_1y_2')|_{x=1} + (x_{n_1}^2 - x_{n_2}^2)\int_0^1 xy_1(x)y_2(x)dx = 0.$$

2) Find the solution of the vibrating membrane problem (i.e., the edges are fixed) in the case where $u(\rho,\varphi,0) = 0$ and $u_t(\rho,\varphi,0) = 1$, $0 < \rho < a$.
**Solution:** $u(\rho,\varphi,t) = \frac{2a}{c}\sum_{n=1}^{\infty}\frac{J_0(\rho x_n/a)}{x_n^2 J_1(x_n)}\sin\frac{ctx_n}{a}$, $J_0(x_n) = 0$.

3) Find the solution of the vibrating membrane problem in the case where $u(\rho,\varphi,0) = 0$ and $u_t(\rho,\varphi,0) = a^2 - \rho^2$, $0 < \rho < a$.
**Solution:** To consider the initial conditions, we need to compute the Fourier-Bessel series of $a^2-\rho^2$. To this end, we begin with writing $a^2 - \rho^2 = \sum_{n=1}^{\infty} A_n J_0(\rho x_n/a)$ (The expansion $a^2 - \rho^2 = \sum_{n=1}^{\infty} B_n J_0(\rho x_n)$ is possible but $J_0(\rho x_n/a)$ is desired because this Bessel function shows up in the general solution). By defining $x = \rho/a$, we have $1 - x^2 = \sum_{n'=1}^{\infty}(A_{n'}/a^2)J_0(xx_{n'})$. Thus we obtain $\int_0^1(1-x^2)J_0(xx_n)xdx = \sum_{n'=1}^{\infty}(A_{n'}/a^2)\int_0^1 J_0(xx_{n'})J_0(xx_n)xdx$. For the left-hand side we introduce $t = xx_n$, and we have $(1/x_n^4)\int_0^{x_n}(x_n^2 - t^2)J_0(t)tdt$. We note that $tJ_0(t) = \frac{d}{dt}[tJ_1(t)]$ and $J_0'(t) = -J_1(t)$. By integration by parts we obtain $\int_0^{x_n}(x_n^2-t^2)J_0(t)tdt = 4x_nJ_1(x_n)$. In the end, we obtain $u(\rho,\varphi,t) = \frac{8a^3}{c}\sum_{n=1}^{\infty}\frac{J_0(\rho x_n/a)}{x_n^4 J_1(x_n)}\sin\frac{ctx_n}{a}$, $J_0(x_n) = 0$.

4) Find the solution of the heat equation $u_t = K\nabla^2 u$ in the infinite cylinder $0 \leq \rho < \rho_{\max}$ satisfying the boundary condition $u(\rho_{\max},\varphi,t) = 0$ and the initial condition $u(\rho,\varphi,0) = \rho_{\max}^2 - \rho^2$.
**Solution:** $u(\rho,\varphi,t) = 8\rho_{\max}^2\sum_{n=1}^{\infty}\frac{J_0(\rho x_n/\rho_{\max})}{x_n^3 J_1(x_n)}e^{-x_n^2Kt/\rho_{\max}^2}$, where $J_0(x_n) = 0$.

5) Find the solution of the heat equation $u_t = K\nabla^2 u + \sigma$ in the infinite cylinder $0 \leq \rho < \rho_{\max}$ satisfying the boundary condition $u(\rho_{\max},\varphi,t) = T_1$ and the initial condition $u(\rho,\varphi,0) = T_2(1 - \rho^2/\rho_{\max}^2)$. Here $K$, $\sigma$, $T_1$, $T_2$ are positive constants.
**Solution:** $u(\rho,\varphi,t) = T_1 + \frac{\sigma(\rho_{\max}^2-\rho^2)}{4K} + \sum_{n=1}^{\infty}A_nJ_0\left(\frac{\rho x_n}{\rho_{\max}}\right)e^{-x_n^2Kt/\rho_{\max}^2}$, where $J_0(x_n) = 0$, $A_n = \frac{8[T_2 - \sigma\rho_{\max}^2/4K]}{x_n^3 J_1(x_n)} - \frac{2T_1}{x_nJ_1(x_n)}$.