Problem Set 7 (2/26, 2/28, 3/10)
Due on Fri, Mar 14

1) Solve the initial-value problem for the heat equation $u_t = K\nabla^2 u$ in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u(0, y, t) = 0$, $u_x(L_1, y, t) = 0$, $u(x, 0, t) = 0$, $u_y(x, L_2, t) = 0$ and the initial condition $u(x, y, 0) = 1$. Find the relaxation time.
Solution: $u(x, y, t) = \frac{4}{\pi^2}\sum_{m,n=1}^{\infty} u_{mn}(x, y, t)$, where $u_{mn}(x, y, t) = \frac{\sin[(m-\frac{1}{2})(\pi x/L_1)]}{m-\frac{1}{2}} \frac{\sin[(n-\frac{1}{2})(\pi y/L_2)]}{n-\frac{1}{2}} e^{-\lambda_{mn}Kt}$. Here $\lambda_{mn} = \left(m - \frac{1}{2}\right)^2 (\pi/L_1)^2 + \left(n - \frac{1}{2}\right)^2 (\pi/L_2)^2$. The relaxation time $\tau = \frac{4}{\pi^2 K}\frac{L_1^2 L_2^2}{L_1^2+L_2^2}$.
2) Choose one set of m, n and time t for $u_{mn}(x, y, t)$ in the solution of 1). Make 3D plots of $u_{mn}(x, y, t)$.† Submit one figure.
3) Solve Laplace's equation $\nabla^2 u = 0$ in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u_x(0, y) = 0$, $u_x(L_1, y) = 0$, $u(x, 0) = T_1$, $u(x, L_2) = T_2$, where T_1 and T_2 are constants.
Solution: $u(x, y) = yT_2/L_2 + (L_2 - y)T_1/L_2$.
4) Solve the initial-value problem for the vibrating membrane in the square $0 < x < L$, $0 < y < L$ with $u(x, y, 0) = 3\sin(\pi x/L)\sin(2\pi y/L) + 4\sin(3\pi x/L)\sin(5\pi y/L)$, $u_t(x, y, 0) = 0$.
Solution: $u(x, y, t) = 3\sin(\pi x/L)\ \sin(2\pi y/L)\ \cos(\pi ct\sqrt{5}/L) + 4\sin(3\pi x/L)$ $\sin(5\pi y/L)\ \cos(\pi ct\sqrt{34}/L)$.
5) Look at the solution $u(x, y, t)$ of 5). Set $t = 0$. Make a 3D plot of $u(x, y, 0)$ and submit it.† Choose $t > 0$. Make 3D plots of $u(x, y, t)$. Submit one figure.
6) Consider cylindrical coordinates (ρ, φ, z). Compute (a) $\nabla^2(\rho^4 \cos 2\varphi)$, (b) $\nabla^2(\rho^2 \cos 2\varphi)$, and (c) $\nabla^2(\rho^n)$, $n = 1, 2, \cdots$. Are these functions smooth in (x, y)?
Solution: (a) $12\rho^2 \cos 2\varphi$; smooth. (b) 0; smooth. (c) $n^2\rho^{n-2}$; smooth if n is even.
7) Find the solution of the equation $\nabla^2 f(\rho) = 0$ satisfying the boundary conditions $f(1) = 3$, $f(2) = 5$.
Solution: $f(\rho) = \frac{2}{\ln 2}\ln\rho + 3$.
8) Find the solution $u(\rho, \varphi)$ of Laplace's equation in the cylindrical region $1 < \rho < 2$ satisfying the boundary conditions $u(1, \varphi) = 0$, $u(2, \varphi) = 0$ for $-\pi < \varphi < 0$ and $u(2, \varphi) = 1$ for $0 < \varphi < \pi$.
Solution: $u(\rho, \varphi) = \frac{\ln\rho}{2\ln 2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\rho^n-\rho^{-n}}{2^n-2^{-n}}\left[\frac{1-(-1)^n}{n}\right]\sin n\varphi$.

† You can use any languages C, C++, Matlab, Python, Mathematica, Maple, Fortran, etc. If you have no idea how to make figures, read §2.5.6 of the textbook and/or email `mmachida@umich.edu`.