Problem Set 6 (2/19, 2/21, 2/24)
Due on Fri, Feb 28

1) Find the relaxation time for Problem 5 in the Homework Problem Set 5.
Solution: We know $u(z,t) = T_1 + \Phi_2 z + \sum_{n=1}^{\infty} A_n \sin \frac{(n-1/2)\pi z}{L} e^{-\left[\frac{(n-1/2)\pi}{L}\right]^2 Kt}$, where $A_n = \frac{2(T_3-T_1)}{(n-1/2)\pi} - \frac{2L\Phi_2(-1)^{n+1}}{(n-1/2)^2\pi^2}$. Hence $\tau = \frac{4L^2}{\pi^2 K}$.
2) Let us consider heat flow in a circular ring of circumference L.
   (a) Find all of the separated solutions of the heat equation $u_t = Ku_{zz}$ $(K > 0)$ satisfying the periodic boundary conditions $u(0,t) = u(L,t)$, $u_z(0,t) = u_z(L,t)$.
   (b) Solve the heat equation $u_t = Ku_{zz}$ $(K > 0)$ satisfying the periodic boundary conditions $u(0,t) = u(L,t)$, $u_z(0,t) = u_z(L,t)$, and the initial conditions $u(z,0) = 100$ if $0 < z < L/2$ and $u(z,0) = 0$ if $L/2 < z < L$.

Solution: (a) $u_n(z,t) = \left(A_n \cos \frac{2n\pi z}{L} + B_n \sin \frac{2n\pi z}{L}\right) \exp\left[-\left(\frac{2n\pi}{L}\right)^2 Kt\right]$, $n = 0, 1, 2, \ldots$. (b) $u(z,t) = 50 + \frac{100}{\pi} \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n}\right] \sin \frac{2n\pi z}{L} \exp\left[-\left(\frac{2n\pi}{L}\right)^2 Kt\right]$.
3) Find the solution of the nonhomogeneous heat equation

$$u_t = Ku_{zz} + ve^{-at} \sin \frac{\pi z}{L}, \quad 0 < z < L,\ t > 0,$$

with $u(0,t) = u(L,t) = u(z,0) = 0$. Here a, v, K are positive constants.
Solution: If $a \neq \pi^2 K/L^2$, then $u(z,t) = -v \sin(\pi z/L) \frac{e^{-at} - e^{-\pi^2 Kt/L^2}}{a - (\pi^2 K/L^2)}$. If $a = \pi^2 K/L^2$, then $u(z,t) = v \sin(\pi z/L) t e^{-\pi^2 Kt/L^2}$.
4) The energy of a vibrating string of tension T_0 and density $\rho = m/L$ is defined by

$$E = \frac{1}{2} \int_0^L \left(\rho y_t^2 + T_0 y_s^2\right) ds.$$

Let

$$y(s,t) = \sum_{n=1}^{\infty} \left(\tilde{A}_n \cos \omega_n t + \tilde{B}_n \sin \omega_n t\right) \sin \frac{n\pi s}{L}$$

be a solution of the wave equation with $\omega_n = n\pi c/L$, where $c^2 = T_0/\rho$. Show that E is independent of t (conservation of energy) by using Parseval's theorem to write E as an infinite series involving $\tilde{A}_n$, $\tilde{B}_n$.
Solution: $E = \frac{L}{4} \sum_{n=1}^{\infty} \left[\rho \omega_n^2 \tilde{B}_n^2 + T_0 \left(\frac{n\pi}{L}\right)^2 \tilde{A}_n^2\right]$.
5) Consider the following initial-value problem for the wave equation $y_{tt} = c^2 y_{ss}$ for $t > 0$, $0 < s < L$ with $y(0,t) = y(L,t) = 0$ for $t > 0$ and $y(s,0) = 0$, $y_t(s,0) = 1$ for $0 < s < L$. Find the Fourier representation of the solution.
Solution: $y(s,t) = \frac{2L}{\pi^2 c} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \sin \frac{n\pi s}{L} \sin \frac{n\pi ct}{L}$.