Winter 2014 Math 454

Problem Set 5 (2/12, 2/14, 2/17)
Due on Fri, Feb 21

1) Prove the orthogonality relations (n, m are nonnegative)
(a) $\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0\ (n \neq m),\ \frac{L}{2}\ (n = m \neq 0),\ L\ (n = m = 0),$
(b) $\int_0^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{2Ln}{\pi(n^2-m^2)}$ ($n+m$ is odd), 0 (otherwise).
Solution: Recall $\cos(a \pm b) = \cos a\cos b \mp \sin a \sin b$ and $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$. From these relations, we can derive the trigonometric identities: $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$, $\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$, and $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$. By using these identities, we can carry out the integrals in the orthogonality relations.

2) Find the eigenvalues and eigenfunctions for the Sturm-Liouville eigenvalue problem

$$\phi''(x) + \lambda\phi(x) = 0, \quad \phi(0) = 0, \quad \phi'(L) = 0.$$

Solution: $\phi_n(x) = A\sin\left((n-\frac{1}{2})\pi x/L\right)$, $\lambda_n = \left((n-\frac{1}{2})\pi/L\right)^2$, $n = 1, 2, \ldots$, and A is an arbitrary constant.

3) Find the eigenvalues and eigenfunctions for the Sturm-Liouville eigenvalue problem

$$\phi''(x) + \lambda\phi(x) = 0, \quad \phi'(0) = 0, \quad \phi(L) = 0.$$

Solution: $\phi_n(x) = A\cos\left((n-\frac{1}{2})\pi x/L\right)$, $\lambda_n = \left((n-\frac{1}{2})\pi/L\right)^2$, $n = 1, 2, \ldots$, and A is an arbitrary constant.

4) Find the eigenvalues and eigenfunctions for the Sturm-Liouville eigenvalue problem

$$\phi''(x) + \lambda\phi(x) = 0, \quad \phi(0) = \phi(L), \quad \phi'(0) = \phi'(L).$$

Solution: $\phi_n(x) = A\cos(2n\pi x/L) + B\sin(2n\pi x/L)$, $\lambda_n = (2n\pi/L)^2$, $n = 1, 2, \ldots$, and $\phi_0(x) = C$, $\lambda_0 = 0$, where A, B, C are arbitrary constants. (The orthogonality theorem can be extended to the case of periodic boundary conditions.)

5) Solve the initial-value problem for the heat equation $u_t = Ku_{zz}$ with the boundary conditions $u(0,t) = T_1$, $u_z(L,t) = \Phi_2$ and the initial condition $u(z,0) = T_3$, where K, T_1, Φ_2, T_3 are positive constants.
Solution: $u(z,t) = T_1 + \Phi_2 z + \sum_{n=1}^{\infty} A_n \sin\frac{(n-1/2)\pi z}{L}\exp\left\{-\left[\frac{(n-1/2)\pi}{L}\right]^2 Kt\right\}$, where $A_n = \frac{2(T_3-T_1)}{(n-1/2)\pi} - \frac{2L\Phi_2(-1)^{n+1}}{(n-1/2)^2\pi^2}$.