Problem Set 4 (1/29, 1/31, 2/3, 2/7, 2/10)
Due on Fri, Feb 14

1) Find the steady-state solution of the heat equation $u_t = K\nabla^2 u$ in the slab $0 < z < L$, with boundary conditions $[u_z - h(u - T_0)](x, y, 0) = 0$ and $[u_z + h(u - T_1)](x, y, L) = 0$. Assume that K, h, T_0, T_1 are all positive constants.
Solution: $u(x, y, z) = U(z) = \frac{T_1(1+hz)+T_0[1+h(L-z)]}{2+hL}$.

2) Let us solve the heat equation in the slab $0 < z < L$:

$$\begin{cases} u_t = Ku_{zz} & 0 < z < L,\ t > 0, \\ u(0, t) = u(L, t) = 0 & t > 0, \\ u(z, 0) = 1 & 0 < z < L, \end{cases}$$

where $K > 0$ is the thermal conductivity.
(a) Find the separated solution depending on λ.
(b) Find the general solution which satisfies the boundary conditions.
(c) Find the particular solution which satisfies the initial and boundary conditions.
Solution: (a) For $\lambda > 0$, $u = \left(A\cos\sqrt{\lambda}z + B\sin\sqrt{\lambda}z\right)e^{-\lambda Kt}$, for $\lambda = 0$, $u = (Az + B)$, for $\lambda < 0$, $u = \left(Ae^{\sqrt{-\lambda}z} + Be^{-\sqrt{-\lambda}z}\right)e^{-\lambda Kt}$. (b) $u = \sum_{n=1}^{\infty} A_n \sin(n\pi z/L)e^{-(n\pi/L)^2Kt}$. (c) $u = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi z}{L}e^{-(n\pi/L)^2Kt}$.

3) Solve the initial-value problem $u_t = Ku_{zz}$ $(K > 0)$ for $t > 0$, $0 < z < L$, with the boundary conditions $u(0, t) = u(L, t) = 0$ and the initial condition $u(z, 0) = z$, $0 < z < L$.
Solution: $u(z, t) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi z}{L}\exp\left[-\left(\frac{n\pi}{L}\right)^2 Kt\right]$.

4) Solve the initial-value problem $u_t = Ku_{zz}$ $(K > 0)$ for $t > 0$, $0 < z < L$, with the boundary conditions $u_z(0, t) = u_z(L, t) = 0$ and the initial condition $u(z, 0) = z$, $0 < z < L$.
Solution: $u(z, t) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)\pi z/L]}{(2n-1)^2}\exp\left[-\frac{(2n-1)^2\pi^2Kt}{L^2}\right]$.

5) Let $\varphi_1 = 1$, $\varphi_2 = x$, $\varphi_3 = x^2$ on the interval $0 \le x \le 1$. Find (a) $\langle\varphi_1, \varphi_2\rangle$, (b) $\langle\varphi_1, \varphi_3\rangle$, (c) $||\varphi_1 - \varphi_2||^2$, and (d) $||\varphi_1 + 3\varphi_2||^2$.
Solution: (a) 1/2, (b) 1/3, (c) 1/3, (d) 7.

6) Find the projection of the function $f(x) = \cos^2 x$ on the orthogonal set $(1, \cos x, \cos 2x)$ on the interval $-\pi \le x \le \pi$.
Solution: $\frac{1}{2} + \frac{1}{2}\cos 2x$.