**Problem Set 3** (1/24, 1/27)
**Due on Fri, Jan 31**

1) Find the mean square error for the Fourier series of the function $f(x) = 1$ for $0 < x < \pi$, $f(0) = 0$, and $f(x) = -1$ for $-\pi < x < 0$. Then, show that $\sigma_N^2 = O(N^{-1})$ as $N \to \infty$.
**Solution:** $\sigma_N^2 = \frac{2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{[(-1)^n - 1]^2}{n^2}$. To show $O(N^{-1})$, define $n = 2m - 1$ and replace the summation with $\sum_{m=(N+2)/2}^{\infty}$ or $\sum_{m=(N+3)/2}^{\infty}$ depending on if $N$ is even or odd. Then use integrals to estimate the sum.

2) Find the mean square error for the Fourier series of $f(x) = x^2$, $-\pi \le x \le \pi$. Then, show that $\sigma_N^2 = O(N^{-3})$ as $N \to \infty$.
**Solution:** $\sigma_N^2 = 8 \sum_{n=N+1}^{\infty} \frac{1}{n^4}$.

3) Write out Parseval's theorem for the Fourier series of
   (a) $f(x) = 1$ for $0 < x < \pi$, $f(0) = 0$, and $f(x) = -1$ for $-\pi < x < 0$,
   (b) $f(x) = x^2$, $-\pi \le x \le \pi$.
**Solution:** (a) $\pi^2/8 = 1 + \frac{1}{9} + \frac{1}{25} + \cdots$, and (b) $\pi^4/90 = 1 + \frac{1}{16} + \frac{1}{81} + \cdots$.

4) Verify that the orthogonality relations hold, in the form
$$\int_{-L}^{L} e^{in\pi x/L} e^{-im\pi x/L} dx = 2L\delta_{nm}.$$
**Solution:** Consider two cases: $n \neq m$ and $n = m$.

5) Use the complex form to find the Fourier series of $f(x) = e^x$, $-L < x < L$.
**Solution:** $e^x = \sum_{n=-\infty}^{\infty} (-1)^n \frac{L + in\pi}{L^2 + n^2\pi^2} (\sinh L) \exp\left(\frac{in\pi}{L} x\right)$.

6) Let $0 < r < 1$, $f(x) = 1/(1 - re^{ix})$, $-\pi < x < \pi$. Find the Fourier series of $f$
**Solution:** Expand $f$ as a power series in $r$. Recall $(1 - x)(1 + x + x^2 + \ldots) = 1$. We obtain $f(x) = \sum_{n=0}^{\infty} r^n e^{inx}$.

7) Let $0 \le r < 1$. Use the above problem 6) to derive
$$\frac{1 - r\cos x}{1 + r^2 - 2r\cos x} = 1 + \sum_{n=1}^{\infty} r^n \cos nx,$$
$$\frac{r \sin x}{1 + r^2 - 2r\cos x} = \sum_{n=1}^{\infty} r^n \sin nx.$$
**Solution:** Use Euler's formula and consider the real part and imaginary part. Note that $0 \le r < 1$ in this problem but we had $0 < r < 1$ in 6).