Problem Set 2 (1/15, 1/17, 1/22)
Due on Fri, Jan 24

1) Compute the Fourier series of $f(x) = x^2$, $-L < x < L$.
Solution: $\frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{L}$.

2) Compute the Fourier series of $f(x) = e^x$, $-L < x < L$.
Solution: $\frac{\sinh L}{L}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{\cos(n\pi x/L) - (n\pi/L)\sin(n\pi x/L)}{1+(n\pi/L)^2}\right]$.

3) Compute the Fourier series of $f(x) = \sin^2 2x$, $-\pi < x < \pi$.
Solution: $\frac{1}{2} - \frac{1}{2}\cos 4x$.

4) Prove the orthogonality relations
(a) $\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = 0\ (n \neq m),\ L\ (n = m \neq 0),\ 0\ (n = m = 0)$,
(b) $\int_{-L}^{L} \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L} dx = 0 \quad$ (all n, m).
Solution: Recall $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$ and $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$. From these relations, we can derive the trigonometric identities: $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$, $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$, and $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$. By using these identities, we can carry out the integrals in the orthogonality relations.

5) Which of the following functions are even, odd, or neither? Explain the reason. (a) $f(x) = x^3 - 3x$, (b) $f(x) = x^2 + 4$, (c) $f(x) = \cos 3x$, (d) $f(x) = x^3 - 3x^2$.
Solution: (a): odd; (b),(c): even; (d): neither.

6) (a) Find the Fourier sine series for $f(x) = e^x$, $0 < x < L$.
(b) Find the Fourier cosine series for $f(x) = e^x$, $0 < x < L$.
Solution: (a) We extend $f(x)$ to the interval $-L < x < L$ by defining $f_O(x) = f(x)$ for $0 < x < L$, $-f(-x)$ for $-L < x < 0$, and 0 for $x = 0$. By considering the Fourier series for this odd function f_O, we obtain the Fourier sine series $\frac{2\pi}{L^2}\sum_{n=1}^{\infty} n\left[\frac{1-e^L(-1)^n}{1+(n\pi/L)^2}\right]\sin\frac{n\pi x}{L}$, (b) We extend $f(x)$ to the interval $-L < x < L$ by defining $f_E(x) = f(x)$ for $0 < x < L$, $f(-x)$ for $-L < x < 0$, and 0 for $x = 0$. By considering the Fourier series for this even function f_E, we obtain the Fourier cosine series $\frac{e^L - 1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\left[\frac{(-1)^n e^L - 1}{1+(n\pi/L)^2}\right]\cos\frac{n\pi x}{L}$.