More Solutions for Final

To prepare for the exam, read your notes (and lecture notes on the web site) in addition to the textbook. Go over homework problems and quizzes. I wrote a few more solutions to homework problem sets.

Below are some additional problems from the textbook.

- Exercise 2.2.2
- Exercise 2.2.3
- Exercise 3.1.13
- Exercise 3.1.14
- Exercise 3.3.8
- Exercise 3.3.9
- Exercise 4.2.13
- Exercise 4.2.14
- Exercise 5.1.15
- Exercise 5.1.16
- Exercise 5.2.6.6
- Exercise 8.1.1
- Exercise 8.4.1

Homework Set 7, Problem 8 Find the solution $u(\rho,\varphi)$ of Laplace's equation in the cylindrical region $1<\rho<2$ satisfying the boundary conditions $u(1,\varphi)=0$, $u(2,\varphi)=0$ for $-\pi<\varphi<0$ and $u(2,\varphi)=1$ for $0<\varphi<\pi$.

Solution We solve

$$\nabla^2 u = u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi} = 0.$$

Assuming a solution of the form $u(\rho,\varphi)=R(\rho)\Phi(\varphi)$, we obtain

$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} = 0.$$

By using the separation constant $\lambda = -\Phi''/\Phi$, we have

$$\Phi'' + \lambda\Phi = 0, \quad \Phi(-\pi)=\Phi(\pi), \quad \Phi'(-\pi)=\Phi'(\pi),$$

$$R'' + \frac{1}{\rho}R' - \frac{\lambda}{\rho^2}R = 0.$$

We obtain Φ as

$$\Phi(\varphi) = A_m\cos m\varphi + B_m \sin m\varphi, \quad \lambda = m^2, \quad m = 0,1,2,\ldots.$$

When $m=0$, two linearly independent solutions to $R''+(1/\rho)R'=0$ are $1, \ln\rho$. For $m\neq 0$, two solutions are found as $R(\rho)=\rho^m, \rho^{-m}$. Therefore the general solution is obtained as

$$u(\rho,\varphi) = A_0 + B_0\ln\rho + \sum_{m=1}^{\infty}\rho^m(A_m\cos m\varphi + B_m\sin m\varphi) + \sum_{m=1}^{\infty}\rho^{-m}(C_m\cos m\varphi + D_m\sin m\varphi),$$

where $A_0, B_0, A_m, B_m, C_m, D_m$ are constants.

The boundary condition $u(1,\varphi)=0$ is expressed as

$$A_0 + \sum_{m'=1}^{\infty}(A_{m'}\cos m'\varphi + B_{m'}\sin m'\varphi) + \sum_{m'=1}^{\infty}(C_{m'}\cos m'\varphi + D_{m'}\sin m'\varphi) = 0.$$

By operating $\int_{-\pi}^{\pi} d\varphi\, \cos m\varphi$ and $\int_{-\pi}^{\pi} d\varphi\, \sin m\varphi$, and using orthogonality relations, we obtain

$$A_0 = 0, \quad C_m = -A_m, \quad D_m = -B_m.$$

That is,

$$u(\rho,\varphi) = B_0\ln\rho + \sum_{m=1}^{\infty}(\rho^m - \rho^{-m})(A_m\cos m\varphi + B_m\sin m\varphi).$$

The boundary condition for $u(2,\varphi)$ is expressed as

$$\theta(\varphi) = B_0\ln 2 + \sum_{m'=1}^{\infty}(2^{m'} - 2^{-m'})(A_{m'}\cos m'\varphi + B_{m'}\sin m'\varphi),$$

where $\theta(\varphi) = 0$ for $\varphi < 0$ and $= 1$ for $\varphi > 0$. By operating $\int_{-\pi}^{\pi} d\varphi \cos m\varphi$ and using orthogonality relations, we obtain for $m = 0, 1, 2, \ldots$,

$$\int_0^\pi \cos m\varphi d\varphi = \int_{-\pi}^{\pi} B_0 \ln 2 \cos m\varphi d\varphi + \sum_{m'=1}^{\infty} (2^{m'} - 2^{-m'}) \int_{-\pi}^{\pi} (A_{m'} \cos m'\varphi + B_{m'} \sin m'\varphi) \cos m\varphi d\varphi$$
$$= 2\pi\delta_{m0} B_0 \ln 2 + \pi(1 - \delta_{m0})(2^m - 2^{-m})A_m.$$

Since the left-hand side is $\pi\delta_{m0}$, we obtain

$$B_0 = \frac{1}{2\ln 2}, \qquad A_m = 0.$$

Similarly by operating $\int_{-\pi}^{\pi} d\varphi \sin m\varphi$ and using orthogonality relations, we obtain for $m = 1, 2, \ldots$,

$$\int_0^\pi \sin m\varphi d\varphi = \int_{-\pi}^{\pi} B_0 \ln 2 \sin m\varphi d\varphi + \sum_{m'=1}^{\infty} (2^{m'} - 2^{-m'}) \int_{-\pi}^{\pi} (A_{m'} \cos m'\varphi + B_{m'} \sin m'\varphi) \sin m\varphi d\varphi$$
$$= \pi(2^m - 2^{-m})B_m.$$

Since the left-hand side is $[1 - (-1)^m]/m$, we obtain

$$B_m = \frac{1 - (-1)^m}{\pi m(2^m - 2^{-m})}.$$

Finally we obtain

$$u(\rho, \varphi) = \frac{\ln \rho}{2\ln 2} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\rho^m - \rho^{-m}}{2^m - 2^{-m}} \left[\frac{1 - (-1)^m}{m}\right] \sin m\varphi.$$

Homework Set 8, Problem 3 Find the solution of the vibrating membrane problem in the case where $u(\rho,\varphi,0)=0$ and $u_t(\rho,\varphi,0)=a^2-\rho^2$, $0<\rho<a$.

Solution We will solve

$$\begin{cases} u_{tt}=c^2\Delta u, & t>0, \quad 0<\rho<a, \\ u=0, & t>0, \quad \rho=a, \\ u=0, & t=0, \quad 0<\rho<a, \\ u_t=a^2-\rho^2, & t=0, \quad 0<\rho<a, \end{cases}$$

where $\Delta=\partial_{\rho\rho}+(1/\rho)\partial_\rho+(1/\rho^2)\partial_{\varphi\varphi}$.

Let us write u as $u(\rho,\varphi,t)=R(\rho)\Phi(\varphi)T(t)$. We obtain

$$\frac{T''}{T}=c^2\left[\frac{R''}{R}+\frac{1}{\rho}\frac{R'}{R}+\frac{1}{\rho^2}\frac{\Phi''}{\Phi}\right].$$

We introduce separation constants as $\mu=-\Phi''/\Phi$, $\lambda=(-1/c^2)T''/T$. We obtain

$$\Phi''+\mu\Phi=0, \quad \Phi(-\pi)=\Phi(\pi), \quad \Phi'(-\pi)=\Phi'(\pi),$$

$$T''+\lambda c^2T=0, \quad T(0)=0.$$

$$R''+\frac{1}{\rho}R+\left(\lambda-\frac{\mu}{\rho^2}\right)R=0, \quad R(a)=0.$$

We obtain

$$\Phi(\varphi)=A\cos m\varphi+B\sin m\varphi, \quad \mu=m^2, \quad m=0,1,2,\ldots,$$

$$T(t)=\sin(ct\sqrt{\lambda}),$$

where A,B are constants. Moreover we have

$$R(\rho)=J_m(\rho\sqrt{\lambda}), \quad \sqrt{\lambda}=\frac{x_n^{(m)}}{a}, \quad J_m(x_n^{(m)})=0, \quad x_n^{(m)}>0, \quad n=1,2,\ldots.$$

The general solution is obtained as

$$u(\rho,\varphi,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A_{mn}\cos m\varphi+B_{mn}\sin m\varphi)\sin\frac{ctx_n^{(m)}}{a}.$$

Let us write $x=\rho/a$. The initial condition for u_t is then written as

$$a^2(1-x^2)=\sum_{m'=0}^{\infty}\sum_{n'=1}^{\infty}\frac{cx_{n'}^{(m')}}{a}J_{m'}(xx_{n'}^{(m')})(A_{m'n'}\cos m'\varphi+B_{m'n'}\sin m'\varphi).$$

Note that for $m,m'=1,2,\cdots$, we have

$$\int_{-\pi}^{\pi}\cos(m'\varphi)\cos(m\varphi)d\varphi=\int_{-\pi}^{\pi}\sin(m'\varphi)\sin(m\varphi)d\varphi=\pi\delta_{m'm}, \quad \int_{-\pi}^{\pi}\sin(m'\varphi)\cos(m\varphi)d\varphi=0.$$

We multiply $\cos m\varphi$ and integrate over φ:

$$\int_{-\pi}^{\pi} a^2(1-x^2)\cos m\varphi d\varphi = \int_{-\pi}^{\pi} \sum_{m'=0}^{\infty}\sum_{n'=1}^{\infty} \frac{cx_{n'}^{(m')}}{a} J_{m'}(xx_{n'}^{(m')})(A_{m'n'}\cos m'\varphi + B_{m'n'}\sin m'\varphi)\cos m\varphi d\varphi.$$

Using the orthogonality relations we find

$$2\pi a^2(1-x^2)\delta_{m0} = 2\pi\delta_{m0}\sum_{n'=1}^{\infty}\frac{cx_{n'}^{(0)}}{a}J_0(xx_{n'}^{(0)})A_{0n'} + \pi(1-\delta_{m0})\sum_{n'=1}^{\infty}\frac{cx_{n'}^{(m)}}{a}J_m(xx_{n'}^{(m)})A_{mn'}.$$

We then multiply $J_m(xx_n^{(m)})x$ and integrate over x:

$$2\pi a^2\delta_{m0}\int_0^1 (1-x^2)J_m(xx_n^{(m)})xdx = 2\pi\delta_{m0}\int_0^1\sum_{n'=1}^{\infty}\frac{cx_{n'}^{(0)}}{a}A_{0n'}J_0(xx_{n'}^{(0)})J_m(xx_n^{(m)})xdx$$
$$+\pi(1-\delta_{m0})\int_0^1\sum_{n'=1}^{\infty}\frac{cx_{n'}^{(m)}}{a}A_{mn'}J_m(xx_{n'}^{(m)})J_m(xx_n^{(m)})xdx.$$

We note that

$$\int_0^1 J_m(xx_{n'}^{(m)})J_m(xx_n^{(m)})xdx = \frac{1}{2}J_{m+1}(x_n^{(m)})^2\delta_{nn'}.$$

Hence we obtain

$$2\pi a^2\delta_{m0}\int_0^1(1-x^2)J_m(xx_n^{(m)})xdx$$
$$= 2\pi\delta_{m0}\frac{cx_n^{(0)}}{a}A_{0n}\frac{J_1(x_n^{(0)})^2}{2} + \pi(1-\delta_{m0})\frac{cx_n^{(m)}}{a}A_{mn}\frac{J_{m+1}(x_n^{(m)})^2}{2}.$$

We see that $A_{mn} = 0$ if $m \neq 0$. When $m = 0$ we have

$$\int_0^1(1-x^2)J_0(xx_n)xdx = \frac{cx_n}{a^3}A_{0n}\frac{J_1(x_n)^2}{2},$$

where $x_n = x_n^{(0)}$. The left-hand side is calculated as follows.

$$\int_0^1(1-x^2)J_0(xx_n)xdx = \frac{1}{x_n^4}\int_0^{x_n}(x_n^2-t^2)J_0(t)tdt \qquad [t = xx_n]$$
$$= \frac{1}{x_n^4}\left\{x_n^2 tJ_1\Big|_0^{x_n} - \left[t^2(tJ_1)\Big|_0^{x_n} - 2\int_0^{x_n}t^2J_1(t)dt\right]\right\} \qquad [tJ_0 = (tJ_1)']$$
$$= \frac{-2}{x_n^4}\int_0^{x_n}t^2(-J_1(t))dt$$
$$= \frac{-2}{x_n^4}\left[t^2J_0\Big|_0^{x_n} - 2\int_0^{x_n}J_0tdt\right] \qquad [J_0' = -J_1]$$
$$= \frac{4}{x_n^4}\int_0^{x_n}J_0(t)tdt$$
$$= \frac{4}{x_n^4}tJ_1(t)\Big|_0^{x_n} \qquad [tJ_0 = (tJ_1)']$$
$$= \frac{4}{x_n^3}J_1(x_n).$$

Therefore we obtain

$$A_{0n} = \frac{8a^3}{cx_n^4 J_1(x_n)}.$$

Similarly by multiplying $\sin m\varphi$ and integrating over φ, we find $B_{mn} = 0$ for all m, n. Finally we obtain

$$u(\rho, \varphi, t) = \frac{8a^3}{c} \sum_{n=1}^{\infty} \frac{J_0(\rho x_n^{(0)}/a)}{[x_n^{(0)}]^4 J_1(x_n^{(0)})} \sin \frac{ctx_n^{(0)}}{a}.$$

Alternative Solution Since it is clear from the equation that u does not depend on φ, we can solve the problem as follows. However the above method works even if initial conditions depend on φ.

Let us write u as $u(\rho, \varphi, t) = u(\rho, t) = R(\rho)T(t)$. We obtain

$$\frac{T''}{T} = c^2 \left[\frac{R''}{R} + \frac{1}{\rho} \frac{R'}{R} \right].$$

We introduce a separation constant as $\lambda = (-1/c^2)T''/T$. We obtain

$$T'' + \lambda c^2 T = 0, \quad T(0) = 0 \quad \Rightarrow \quad T(t) = \sin(ct\sqrt{\lambda}).$$

Moreover we have

$$R'' + \frac{1}{\rho} R + \lambda R = 0, \quad R(a) = 0.$$

$$\Rightarrow \quad R(\rho) = J_0(\rho\sqrt{\lambda}), \quad \sqrt{\lambda} = \frac{x_n}{a}, \quad J_0(x_n) = 0, \quad x_n > 0, \quad n = 1, 2, \ldots.$$

The general solution is obtained as

$$u(\rho, t) = \sum_{n=1}^{\infty} A_n J_0 \left(\frac{\rho x_n}{a} \right) \sin \frac{ctx_n}{a},$$

where A_n are constants. Let us write $x = \rho/a$. The initial condition for u_t is then written as

$$a^2(1 - x^2) = \sum_{n'=1}^{\infty} A_{n'} \frac{cx_{n'}}{a} J_0(xx_{n'}).$$

We multiply $J_0(xx_n)x$ and integrate over x:

$$a^2 \int_0^1 (1 - x^2) J_0(xx_n) x dx = \int_0^1 \sum_{n'=1}^{\infty} A_{n'} \frac{cx_{n'}}{a} J_0(xx_{n'}) J_0(xx_n) x dx.$$

Thus we arrive at

$$\int_0^1 (1 - x^2) J_0(xx_n) x dx = A_n \frac{cx_n}{a^3} \frac{J_1(x_n)^2}{2}.$$

Homework Set 9, Problem 2 Solve $\nabla^2[f(r)] = -1$ with the boundary condition $f(a) = 0$ and $f(0)$ finite.

Solution We note that

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta f_\theta) + \frac{1}{r^2\sin^2\theta} f_{\varphi\varphi}.$$

Therefore,

$$\nabla^2[f(r)] = -1 \quad \Leftrightarrow \quad \frac{1}{r^2}(r^2 f')' = -1 \quad \Leftrightarrow \quad f'' + \frac{2}{r}f' = -1.$$

We find a solution $f_c(r)$ (a complementary function) to the homogeneous equation,

$$f_c'' + \frac{2}{r}f_c' = 0.$$

By assuming the form $f_c = r^k$, we obtain $k(k+1) = 0$ or $k = 0, -1$. Hence we have

$$f_c(r) = A + \frac{B}{r},$$

where A, B are constants. Furthermore using the method of undetermined coefficients we assume a particular solution of the form $f_p(r) = Cr^2$. By substituting this f_p for f in $f'' + \frac{2}{r}f' = -1$, we find

$$C = -\frac{1}{6}.$$

Thus a general solution is written as

$$f(r) = f_c(r) + f_p(r) = A + \frac{B}{r} - \frac{r^2}{6}.$$

Since $f(0)$ is finite, we can set $B = 0$. To satisfy $f(a) = 0$, we must choose $A = a^2/6$. Therefore we obtain

$$f(r) = \frac{a^2 - r^2}{6}.$$

Homework Set 10, Problem 4 Find the Fourier transform of $f(x) = 1/[1 + (x-3)^2]$.

Solution First we calculate $g(x) = \int_{-\infty}^{\infty} e^{-|\mu|} e^{i\mu x} d\mu$.

$$g(x) = \int_{-\infty}^{\infty} e^{-|\mu|} e^{i\mu x} d\mu = \int_{0}^{\infty} e^{-\mu} e^{i\mu x} d\mu + \int_{-\infty}^{0} e^{\mu} e^{i\mu x} d\mu = \int_{0}^{\infty} e^{-(1-ix)\mu} d\mu + \int_{0}^{\infty} e^{-(1+ix)\mu} d\mu$$
$$= \frac{1}{1-ix} + \frac{1}{1+ix} = \frac{2}{1+x^2}.$$

The result implies

$$e^{-|\mu|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\mu x} dx.$$

Noting the inverse Fourier transform $\tilde{f}(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\mu x} dx$, we obtain

$$\tilde{f}(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+(x-3)^2} e^{-i\mu x} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+x'^2} e^{-i\mu(x'+3)} dx'$$
$$= \frac{e^{-3i\mu}}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+x^2} e^{-i\mu x} dx = \frac{e^{-3i\mu}}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-i\mu x} dx$$
$$= \frac{1}{2} e^{-3i\mu} e^{-|\mu|}.$$

Alternative Solution We can also find $\tilde{f}(\mu)$ directly.

$$\tilde{f}(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+(x-3)^2} e^{-i\mu x} dx = \frac{1}{2\pi} e^{-3i\mu} \int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-i\mu x} dx$$
$$= \frac{1}{4\pi} e^{-3i\mu} \left[\int_{-\infty}^{\infty} \frac{e^{-i\mu x}}{1-ix} dx + \int_{-\infty}^{\infty} \frac{e^{-i\mu x}}{1+ix} dx \right]$$
$$= \frac{1}{4\pi} e^{-3i\mu} \left[e^{-\mu} \int_{-\infty}^{\infty} \frac{e^{(1-ix)\mu}}{1-ix} dx + e^{\mu} \int_{-\infty}^{\infty} \frac{e^{-(1+ix)\mu}}{1+ix} dx \right]$$
$$= \frac{1}{4\pi} e^{-3i\mu} \left[e^{-\mu} \int_{-\infty}^{\infty} \int_{-\infty}^{\mu} e^{(1-ix)t} dt dx + e^{\mu} \int_{-\infty}^{\infty} \int_{\mu}^{\infty} e^{-(1+ix)t} dt dx \right]$$
$$= \frac{1}{2} e^{-3i\mu} \left[e^{-\mu} \int_{-\infty}^{\mu} e^{t} \int_{-\infty}^{\infty} \frac{e^{-ixt}}{2\pi} dx dt + e^{\mu} \int_{\mu}^{\infty} e^{-t} \int_{-\infty}^{\infty} \frac{e^{-ixt}}{2\pi} dx dt \right]$$
$$= \frac{1}{2} e^{-3i\mu} \left[e^{-\mu} \int_{-\infty}^{\mu} e^{t} \delta(t) dt + e^{\mu} \int_{\mu}^{\infty} e^{-t} \delta(t) dt \right]$$
$$= \frac{1}{2} e^{-3i\mu} e^{-|\mu|}.$$

Homework Set 11, Problem 4 Find the solution of the heat equation $u_t - Ku_{xx} = h$ for $0 < x < \infty$ satisfying the boundary conditions $u_x(0,t) = 0$, $u(x,0) = 0$.

Solution Let us start by recalling that $u_t - Ku_{xx} = h$, $-\infty < x < \infty$ with the initial condition $u(x,0) = 0$ is solved as

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,x',t-s)h(x',s)dx'ds,$$

where $G(x,x',t) = \frac{1}{\sqrt{4\pi Kt}} e^{-(x-x')^2/4Kt}$ is the heat kernel.

We extend h as

$$h_E(x,t) = \begin{cases} h(x,t) & x > 0, \\ h(-x,t) & x < 0. \end{cases}$$

Then we have

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,x',t-s)h_E(x',s)dx'ds.$$

We obtain

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^{\infty} G(x,x',t-s)h(x',s)dx'ds + \int_0^t \int_{-\infty}^{0} G(x,x',t-s)h(-x',s)dx'ds \\ &= \int_0^t \int_0^{\infty} G(x,x',t-s)h(x',s)dx'ds + \int_0^t \int_0^{\infty} G(x,-x',t-s)h(x',s)dx'ds \\ &= \int_0^t \int_0^{\infty} \left[G(x,x',t-s) + G(x,-x',t-s)\right] h(x',s)dx'ds. \end{aligned}$$

Therefore,

$$u(x,t) = \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi K(t-s)}} \left[e^{-(x-x')^2/4K(t-s)} + e^{-(x+x')^2/4K(t-s)}\right] h(x',s)dx'ds.$$