More Solutions for Midterm 2

To prepare for the exam, read your notes (and lecture notes on the web site) in addition to the textbook. Go over homework problems and quizzes. I wrote a few more solutions to homework problem sets.

Homework Set 4, Problem 3 Solve the initial-value problem $u_t = Ku_{zz}$ $(K > 0)$ for $t > 0$, $0 < z < L$, with the boundary conditions $u(0,t) = u(L,t) = 0$ and the initial condition $u(z,0) = z$, $0 < z < L$.

Solution Assuming the form $u(z,t) = \phi(z)T(t)$, we have

$$\phi'' + \lambda\phi = 0, \quad 0 < z < L, \quad \phi(0) = \phi(L) = 0, \qquad T' + \lambda KT = 0, \quad t > 0.$$

We obtain

$$\phi(z) = \phi_n(z) = \sin\frac{n\pi z}{L}, \quad \lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \cdots,$$

and $T(t) = e^{-\lambda Kt} = e^{-\lambda_n Kt}$. To take into account the initial condition, we write

$$u(z,t) = \sum_{n=1}^{\infty} A_n \phi_n(z) e^{-\lambda_n Kt}.$$

Using the orthogonality for ϕ_n, the coefficients A_n are determined as

$$\begin{aligned} A_n &= \frac{\int_0^L z\phi_n(z)dz}{\int_0^L \phi_n(z)^2 dz} = \frac{\int_0^L z\sin(n\pi z/L)dz}{L/2} \\ &= \frac{2}{L}\left[\frac{-L}{n\pi} z\cos\frac{n\pi z}{L}\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi z}{L}dz\right] \\ &= \frac{2L}{n\pi}(-1)^{n+1}. \end{aligned}$$

Finally, the solution is obtained as

$$u(z,t) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt}.$$

Homework Set 5, Problem 5 Solve the initial-value problem for the heat equation $u_t = Ku_{zz}$ with the boundary conditions $u(0,t) = T_1$, $u_z(L,t) = \Phi_2$ and the initial condition $u(z,0) = T_3$, where K, T_1, Φ_2, T_3 are positive constants.

Solution First we note that $U''(z) = 0$, $U(0) = T_1$, $U'(L) = \Phi_2$ is solved as $U(z) = T_1 + \Phi_2 z$. We also note the integrals

$$\int_a^b \sin^2 kx\, dx = \frac{1}{2}\left[x - \frac{1}{2k}\sin 2kx\right]_a^b, \quad \int_a^b x\sin kx\, dx = \left[-\frac{x}{k}\cos kx + \frac{1}{k^2}\sin kx\right]_a^b.$$

Define $v(z,t) = u(z,t) - U(z)$. This function obeys $v_t = Kv_{zz}$, $t > 0$, $0 < z < L$ with boundary conditions $v(0,t) = 0$ and $v_z(L,t) = 0$, $t > 0$, and initial condition $u(z,0) = T_3 - U(z)$, $0 < z < L$. Assume the form $v(z,t) = \phi(z)T(t)$. We have

$$\phi'' + \lambda\phi = 0, \quad 0 < z < L, \quad \phi(0) = 0, \quad \phi'(L) = 0, \qquad T' + \lambda KT = 0, \quad t > 0.$$

Two linearly independent functions $e^{\sqrt{-\lambda}z}$ and $e^{-\sqrt{-\lambda}z}$ (z and 1 for $\lambda = 0$) satisfy $\phi'' + \lambda\phi = 0$. Taking the boundary condition into account, we look for solutions of the forms $a\cos\sqrt{\lambda}z + b\sin\sqrt{\lambda}z$ ($\lambda > 0$), $Az + B$ ($\lambda = 0$), and $Ae^{\sqrt{-\lambda}z} + Be^{-\sqrt{-\lambda}z}$ ($\lambda < 0$). Nontrivial solutions are found as

$$\lambda_n = \left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2, \quad \phi_n(z) = \sin\sqrt{\lambda_n}z, \quad n = 1, 2, \cdots.$$

We also obtain $T(t) = e^{-\lambda Kt} = e^{-\lambda_n Kt}$. To find the solution satisfying the initial condition, we write

$$v(z,t) = \sum_{n=1}^{\infty} A_n\phi_n(z)e^{-\lambda_n Kt}.$$

Using the orthogonality of ϕ_n, the coefficients A_n are determined as

$$A_n = \frac{\int_0^L [T_3 - U(z)]\,\phi_n(z)dz}{\int_0^L \phi_n(z)^2 dz} = \frac{\int_0^L (T_3 - T_1 - \Phi_2 z)\sin(\sqrt{\lambda_n}z)dz}{L/2} = \frac{2}{L}\left[\frac{T_3 - T_1}{\sqrt{\lambda_n}} - \frac{\Phi_2}{\lambda_n}(-1)^{n+1}\right].$$

Finally, the solution is obtained as

$$u(z,t) = U(z) + \sum_{n=1}^{\infty} A_n\phi_n(z)e^{-\lambda_n Kt} = T_1 + \Phi_2 z + \sum_{n=1}^{\infty} A_n \sin\frac{\left(n-\frac{1}{2}\right)\pi z}{L} e^{-\left[\frac{(n-1/2)\pi}{L}\right]^2 Kt},$$

where

$$A_n = \frac{2(T_3 - T_1)}{(n-1/2)\pi} - \frac{2L\Phi_2(-1)^{n+1}}{(n-1/2)^2\pi^2}.$$

Homework Set 6, Problem 3 Find the solution of the nonhomogeneous heat equation

$$u_t = Ku_{zz} + ve^{-at}\sin\frac{\pi z}{L}, \quad 0 < z < L,\ t > 0,$$

with $u(0,t) = u(L,t) = u(z,0) = 0$. Here a, v, K are positive constants.

Solution We write

$$R(z,t) = ve^{-at}\sin\frac{\pi z}{L}.$$

We consider a Sturm-Liouville eigenvalue problem

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = \phi(L) = 0.$$

Note that we imposed the same boundary conditions as $u(z,t)$ satisfies. We obtain

$$\phi(z) = \phi_n(z) = \sin\frac{n\pi z}{L}, \quad \lambda = \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots.$$

We then expand u, R with ϕ_n.

$$u(z,t) = \sum_{n=1}^{\infty} u_n(t)\phi_n(z), \quad R(z,t) = \sum_{n=1}^{\infty} R_n(t)\phi_n(z).$$

By plugging these into the heat equation, we obtain

$$u_n'(t) = -\lambda_n K u_n(t) + R_n(t),$$

where $u_n(0) = 0$ from the initial condition and $R_n(t)$ is obtained as

$$R_n(t) = \frac{\int_0^L R(z,t)\phi_n(z)dz}{\int_0^L \phi_n(z)^2 dz} = \frac{2}{L}\int_0^L (ve^{-at})\phi_1(z)\phi_n(z)dz = ve^{-at}\delta_{n1}.$$

We have

$$u_1' + \lambda_1 K u_1 = ve^{-at} \quad \Rightarrow \quad \frac{d}{dt}\left(u_1(t)e^{\lambda_1 Kt}\right) = ve^{(\lambda_1 K - a)t}$$

$$\Rightarrow \quad u_1(t) = ve^{-\lambda_1 Kt}\int_0^t e^{(\lambda_1 K - a)s}ds = -v\frac{e^{-at} - e^{-\lambda_1 Kt}}{a - \lambda_1 K}.$$

Otherwise for $n \geq 2$, we obtain $u_n' + \lambda_1 K u_n = 0 \Rightarrow u_n(t) = 0$. Therefore we obtain

$$u(z,t) = u_1(t)\phi_1(z) = -v\frac{e^{-at} - e^{-(\pi/L)^2 Kt}}{a - (\pi/L)^2 K}\sin\frac{\pi z}{L}.$$

We note that we have $\frac{d}{dt}\left(u_1 e^{\lambda_1 Kt}\right) = v$ if $a = \lambda_1 K = \pi^2 K/L^2$. In this case, we obtain

$$u(z,t) = u_1(t)\phi_1(z) = vte^{-(\pi/L)^2 Kt}\sin\frac{\pi z}{L}.$$

Homework Set 7, Problem 3 Solve Laplace's equation $\nabla^2 u = 0$ in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u_x(0, y) = 0$, $u_x(L_1, y) = 0$, $u(x, 0) = T_1$, $u(x, L_2) = T_2$, where T_1 and T_2 are constants.

Solution If we write $u(x, y) = \phi_1(x)\phi_2(y)$, we can introduce separation constants as $\frac{\phi_1''}{\phi_1} = -\lambda$, $\frac{\phi_2''}{\phi_2} = \lambda$. We obtain

$$\phi_1'' + \lambda\phi_1 = 0, \qquad \phi_1'(0) = \phi_1'(L_1) = 0$$

Thus,

$$\phi_1(x) = \phi_1^{(n)}(x) = \cos\frac{n\pi x}{L_1}, \quad \lambda = \lambda_n \left(\frac{n\pi}{L_1}\right)^2, \quad n = 0, 1, 2, \ldots.$$

Note that $\phi_2 = \phi_2^{(n)}$ is also labeled by n since it depends on λ_n. The solution is expressed as

$$u(x, y) = \sum_{n=0}^{\infty} \phi_1^{(n)}(x)\phi_2^{(n)}(y).$$

We can write the last two boundary conditions as

$$\sum_{n=0}^{\infty} \phi_1^{(n)}(x)\phi_2^{(n)}(0) = T_1, \qquad \sum_{n=0}^{\infty} \phi_1^{(n)}(x)\phi_2^{(n)}(L_2) = T_2.$$

Using the orthogonality relations for $\phi_1^{(n)}(x)$ we obtain

$$\phi_2^{(n)}(0) \int_0^L [\phi_1^{(n)}(x)]^2 dx = T_1 \int_0^L \phi_1^{(n)}(x) dx = T_1 L \delta_{n0},$$

$$\phi_2^{(n)}(L_2) \int_0^L [\phi_1^{(n)}(x)]^2 dx = T_2 \int_0^L \phi_1^{(n)}(x) dx = T_2 \delta_{n0}.$$

That is, $\phi_2^{(n)}(0) = \phi_2^{(n)}(L_2) = 0$ for $n \geq 1$ and $\phi_2^{(0)}(0) = T_1$, $\phi_2^{(0)}(L_2) = T_2$. We have

$$\phi_2'' - \lambda\phi_2 = 0 \quad (\lambda > 0), \quad \phi_2(0) = \phi_2(L_2) = 0 \quad \Rightarrow \quad \phi_2 = 0,$$

and

$$\phi_2'' = 0, \quad \phi_2(0) = T_1, \quad \phi_2(L_2) = T_2 \quad \Rightarrow \quad \phi_2 = T_1 + \frac{T_2 - T_1}{L_2} y.$$

Finally,

$$u(x, y) = \phi_1^{(0)}(x)\phi_2^{(0)}(y) = T_1 + \frac{(T_2 - T_1)}{L_2} y = \frac{T_2}{L_2} y + \frac{T_1}{L_2}(L_2 - y).$$