More Solutions for Midterm 1

To prepare for the exam, read your notes (and lecture notes on the web site) in addition to the textbook. Go over homework problems and quizzes. I wrote a few more solutions to homework problem sets.

Homework Set 1, Problem 7 Find the separated solutions $u(x,t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 < x < L$, $t > 0$, that satisfy the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$.

Solution We look for a separated solution $u(x,t) = X(x)T(t)$. We get

$$\frac{T'}{T} - \frac{X''}{X} = 0.$$

By introducing the separation constant λ, we obtain ‡

$$T'(t) = \lambda T(t), \quad X''(x) = \lambda X(x).$$

Thus, for three cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$, we have §

$$u(x,t) = \begin{cases} (A_1 \cosh(kx) + A_2 \sinh(kx))\, e^{k^2 t} & \text{for } \lambda = k^2,\ k > 0, \\ A_1 x + A_2 & \text{for } \lambda = 0, \\ (A_1 \cos(lx) + A_2 \sin(lx))\, e^{-l^2 t} & \text{for } \lambda = -l^2,\ l > 0. \end{cases}$$

For $\lambda > 0$, the boundary conditions imply $A_1 = A_2 = 0$. Similarly for $\lambda = 0$, we can conclude $A_1 = A_2 = 0$. Only the case $\lambda < 0$ has nontrivial solutions. From the boundary conditions, we have $A_1 = 0$ and $\sin(lL) = 0$. Hence $lL = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Let C be a constant. We obtain

$$u(x,t) = C \sin \frac{n\pi x}{L} e^{-(n\pi/L)^2 t} \quad (n = 1, 2, \ldots).$$

‡ The introduction of λ is not unique. So, $T'(t) = -\lambda T(t)$, $X''(x) = -\lambda X(x)$, and $T'(t) = 2\lambda T(t)$, $X''(x) = 2\lambda X(x)$ are all fine. The final solution $u(x,t)$ will be the same.

§ In the case of $\lambda > 0$, we can also write $\left(A_1 e^{kx} + A_2 e^{-kx}\right) e^{k^2 t}$. In the case of $\lambda < 0$, we can also write $\left(A_1 e^{ilx} + A_2 e^{-ilx}\right) e^{-l^2 t}$.

Homework Set 2, Problem 6(b) Find the Fourier cosine series for $f(x) = e^x$, $0 < x < L$.

Solution We extend $f(x)$ to the interval $-L < x < L$ by defining $\|$

$$f_E(x) = \begin{cases} e^x & \text{for } 0 < x < L, \\ 0 & \text{for } x = 0, \\ e^{-x} & \text{for } -L < x < 0. \end{cases}$$

We consider the Fourier series for this even function f_E. Note that $B_n = 0$. We have

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f_E(x)dx = \frac{1}{2L}\left(\int_0^L e^x dx + \int_{-L}^0 e^{-x}dx\right).$$

By $y = -x$, the second integral is $\int_{-L}^0 e^{-x}dx = \int_L^0 e^y(-dy) = \int_0^L e^y dy$.¶ Hence,

$$A_0 = \frac{1}{L}\int_0^L e^x dx = \frac{e^L - 1}{L}.$$

Similarly,

$$A_n = \frac{1}{L}\int_L^L f_E(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L e^x \cos\frac{n\pi x}{L}dx.$$

By using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$,$^+$

$$A_n = \frac{1}{L}\int_0^L e^x\left(e^{in\pi x/L} + e^{-inx/L}\right)dx = \frac{1}{L}\left(I + \bar{I}\right) = \frac{2}{L}\operatorname{Re} I,$$

where $\bar{I}$ is the complex conjugate of I and

$$I = \int_0^L e^{(1+in\pi/L)x}dx = \left.\frac{e^{(1+in\pi/L)x}}{1+in\pi/L}\right|_0^L = \frac{e^L e^{in\pi} - 1}{1 + in\pi/L} = \frac{1 - in\pi/L}{1 + (n\pi/L)^2}\left(e^L(-1)^n - 1\right).$$

Therefore we obtain

$$A_n = \frac{2}{L}\frac{e^L(-1)^n - 1}{1 + (n\pi/L)^2}.$$

The Fourier cosine series is obtained as

$$e^x = \frac{e^L - 1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\frac{e^L(-1)^n - 1}{1 + (n\pi/L)^2}\cos\frac{n\pi x}{L}.$$

$\|$ In this case, $f_E(0)$ is not necessarily zero.

¶ Since f_E is even, actually we can immediately write down $\frac{1}{2L}\int_{-L}^L f_E(x)dx = \frac{1}{L}\int_0^L f_E(x)dx$.

$^+$ We can also use integration by parts: $\int_0^L e^x \cos\frac{n\pi x}{L}dx = e^x\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L - \frac{L}{n\pi}\int_0^L e^x \sin\frac{n\pi x}{L}dx = -\frac{L}{n\pi}\left(-e^x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L + \frac{L}{n\pi}\int_0^L e^x\cos\frac{n\pi x}{L}dx\right)$. By comparing the leftmost side and the rightmost side, we obtain

$$\int_0^L e^x\cos\frac{n\pi x}{L}dx = \left[1 + \left(\frac{L}{n\pi}\right)^2\right]^{-1}\left(-\frac{L}{n\pi}\right)^2\left(e^L\cos n\pi - 1\right) = \frac{e^L(-1)^n - 1}{1 + (n\pi/L)^2}.$$

Homework Set 3, Problem 3(a) Write out Parseval's theorem for the Fourier series of $f(x) = 1$ for $0 < x < \pi$, $f(0) = 0$, and $f(x) = -1$ for $-\pi < x < 0$.

Solution Since $f(x)$ is an odd function, Parseval's theorem is written as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{1}{2}\sum_{n=1}^{\infty} B_n^2.$$

Here,

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi}\sin(nx)dx = \frac{2}{\pi}\,\frac{-1}{n}\cos(nx)\bigg|_0^{\pi} = \frac{2\left(1-(-1)^n\right)}{n\pi}.$$

Hence, $B_n^2 = 16/(n\pi)^2$ if n is odd and $B_n^2 = 0$ if n is even. On the other hand,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{1}{\pi}\int_0^{\pi} f^2(x)dx = 1.$$

We obtain

$$1 = \frac{1}{2}\left(B_1^2 + B_3^2 + B_5^2 + \cdots\right) = \frac{8}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right).$$

Finally we obtain

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \cdots.$$

Homework Set 3, Problem 3(b) Write out Parseval's theorem for the Fourier series of $f(x) = x^2$, $-\pi \le x \le \pi$.

Solution Since $f(x)$ is an even function, Parseval's theorem is written as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2.$$

Here,

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{\pi^2}{3},$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx = \frac{4(-1)^n}{n^2}.$$

Moreover,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{1}{\pi}\int_0^{\pi} f^2(x)dx = \frac{\pi^4}{5}.$$

We obtain

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right).$$

Finally we obtain

$$\frac{\pi^4}{90} = 1 + \frac{1}{16} + \frac{1}{81} + \cdots.$$