MATH 454 SECTION 002
MIDTERM 2

March 24, 2014, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- Formulae listed at the end can be used without proof.
- Theorems listed at the end can be used without proof.
- You can also use results from other problems, e.g., you can use Problem 1 when you solve Problem 2.
- One side of a US letter size paper ($8.5'' \times 11''$) with notes is OK.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
TOTAL	40	

Problem 1. (10 points) Let us consider the Sturm-Liouville eigenproblem $\phi''(x)+\mu\phi(x)=0$, $\phi(0)=\phi'(L)=0$. The eigenvalues are $\mu=\mu^{(m)}=\left[(m-\frac{1}{2})\pi/L\right]^2$, $m=1,2,\ldots$, and the eigenfunctions are $\phi(x)=\phi^{(m)}(x)=\sin\left(\sqrt{\mu^{(m)}}x\right)$. We consider integrals of $\phi^{(m)}(x)$.

We have $\int_0^L \left[\phi^{(m)}(x)\right]^2 dx = \frac{L}{2}$ and $\int_0^L x\phi^{(m)}(x)dx = \left[\frac{L}{(m-\frac{1}{2})\pi}\right]^2 (-1)^{m+1}$.

Show $\int_0^L \phi^{(m)}(x)dx = \frac{L}{\left(m-\frac{1}{2}\right)\pi}$.

Solution

$$\begin{aligned}\int_0^L \phi^{(m)}(x)dx &= \int_0^L \sin\frac{(m-\frac{1}{2})\pi x}{L}dx \\ &= \frac{-L}{(m-\frac{1}{2})\pi}\cos\frac{(m-\frac{1}{2})\pi x}{L}\Big|_0^L \\ &= \frac{L}{(m-\frac{1}{2})\pi}.\end{aligned}$$

Problem 2. (10 points) Consider the heat equation $u_t = K\nabla^2 u$ in the column $0 < x < L$, $0 < y < L$ with the boundary conditions $u(0, y, t) = 0$, $u_x(L, y, t) = 0$, $u(x, 0, t) = 0$, $u_y(x, L, t) = 0$ and the initial condition $u(x, y, 0) = 0.25$. Find $u(x, y, t)$. You can use theorems listed at the end of this problem set. But state clearly which theorems you use.

Solution If we write $u(x, y, t) = \phi_1(x)\phi_2(y)T(t)$, we can introduce separation constants as $\frac{T'}{T} = -\lambda K$, $\frac{\phi_1''}{\phi_1} = -\mu_1$, $\frac{\phi_2''}{\phi_2} = -\mu_2$, where $\lambda = \mu_1 + \mu_2$. Using Problem 1, we can solve $\phi_1'' + \mu_1\phi_1 = 0$, $\phi_1'(0) = \phi_1(L) = 0$, and $\phi_2'' + \mu_2\phi_2 = 0$, $\phi_2'(0) = \phi_2(L) = 0$ as

$$\phi_1(x) = \sin(\sqrt{\mu_1}x), \quad \mu_1 = \left[\frac{(m-\frac{1}{2})\pi}{L}\right]^2, \quad \phi_2(y) = \sin(\sqrt{\mu_2}y), \quad \mu_2 = \left[\frac{(n-\frac{1}{2})\pi}{L}\right]^2,$$

where $m, n = 1, 2, \ldots$. We can also solve $T' + \lambda K T = 0$ as $T(t) = e^{-\lambda K t}$. Thus the general solution is written as

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\phi_1^{(m)}(x)\phi_2^{(n)}(y)e^{-\lambda_{mn}Kt},$$

where $\lambda_{mn} = \left[(m-\frac{1}{2})\pi/L\right]^2 + \left[(n-\frac{1}{2})\pi/L\right]^2$.

By the initial condition we have

$$\frac{1}{4} = \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'}\phi_1^{(m')}(x)\phi_2^{(n')}(y).$$

We multiply $\phi_1^{(m)}(x)\phi_2^{(n)}(y)$ on both sides and integrate both sides over x, y:

$$\int_0^L\int_0^L \frac{1}{4}\phi_1^{(m)}(x)\phi_2^{(n)}(y)dxdy = \int_0^L\int_0^L \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'}\phi_1^{(m')}(x)\phi_2^{(n')}(y)\phi_1^{(m)}(x)\phi_2^{(n)}(y)dxdy.$$

$$\text{LHS} = \frac{1}{4}\int_0^L \phi_2^{(n)}(y)dy\int_0^L \phi_1^{(m)}(x)dx = \frac{1}{4}\frac{L}{(m-\frac{1}{2})\pi}\frac{L}{(n-\frac{1}{2})\pi},$$

$$\text{RHS} = \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'}\int_0^{L_1} \phi_1^{(m')}(x)\phi_1^{(m)}(x)dx\int_0^{L_2} \phi_2^{(n')}(y)\phi_2^{(n)}(y)dy = B_{mn}\frac{L}{2}\frac{L}{2},$$

where we used $\int_0^L \phi_1^{(m')}(x)\phi_1^{(m)}(x)dx = 0$ $(m' \neq m)$ and $\int_0^L \phi_2^{(n')}(y)\phi_2^{(n)}(y)dy = 0$ $(n' \neq n)$ from Theorem 3 on the last page. Hence $B_{mn} = [(m-\frac{1}{2})\pi]^{-1}[(n-\frac{1}{2})\pi]^{-1}$. Finally we obtain

$$u(x, y, t) = \frac{1}{\pi^2}\sum_{m,n=1}^{\infty}\frac{\sin[(m-\frac{1}{2})(\pi x/L)]}{m-\frac{1}{2}}\frac{\sin[(n-\frac{1}{2})(\pi y/L)]}{n-\frac{1}{2}}e^{-\lambda_{mn}Kt}.$$

(continued)

Remark The orthogonality relations used in this problem

$$\int_0^L \sin\frac{(n-\frac{1}{2})\pi x}{L}\sin\frac{(m-\frac{1}{2})\pi x}{L}dx = 0, \qquad \text{for } n \neq m \quad (n, m = 1, 2, \ldots)$$

are different from the orthogonality relations in Theorem 2. The present orthogonality relations are rather direct consequence of Theorem 3. Let us put $s(x) = 1$, $\rho(x) = 1$, $q(x) = 0$, $a = 0$, $b = L$, $\alpha = \pi/2$, $\beta = 0$ in Theorem 3. We obtain

$$\phi''(x) + \lambda\phi(x) = 0, \quad 0 < x < L, \qquad \phi'(0) = \phi(L) = 0.$$

If we write $\lambda = \lambda_n$, $\phi(x) = \phi^{(n)}(x)$, Theorem 3 states

$$\int_0^L \phi^{(n)}(x)\phi^{(m)}(x)dx = 0 \qquad \text{for} \quad \lambda_n \neq \lambda_m.$$

We can prove this without using explicit form of $\phi(x)$ and λ as follows (see Chapter 1). Suppose $n \neq m$ and $\lambda_n \neq \lambda_m$. We write

$$\phi^{(n)''}(x) + \lambda_n\phi^{(n)}(x) = 0, \qquad \phi^{(m)''}(x) + \lambda_m\phi^{(m)}(x) = 0.$$

By multiplying $\phi^{(m)}(x)$ $(\phi^{(n)}(x))$ and integrating over x, we obtain

$$\int_0^L \phi^{(n)''}(x)\phi^{(m)}(x)dx + \lambda_n\int_0^L \phi^{(n)}(x)\phi^{(m)}(x) = \phi^{(n)'}(x)\phi^{(m)}(x)\Big|_0^L - \int_0^L \phi^{(n)'}(x)\phi^{(m)'}(x)dx + \lambda_n\int_0^L \phi^{(n)}(x)\phi^{(m)}(x)$$

$$= -\int_0^L \phi^{(n)'}(x)\phi^{(m)'}(x)dx + \lambda_n\int_0^L \phi^{(n)}(x)\phi^{(m)}(x) = 0,$$

and similarly

$$\int_0^L \phi^{(m)''}(x)\phi^{(n)}(x)dx + \lambda_m\int_0^L \phi^{(m)}(x)\phi^{(n)}(x)dx = -\int_0^L \phi^{(m)'}(x)\phi^{(n)'}(x)dx + \lambda_m\int_0^L \phi^{(m)}(x)\phi^{(n)}(x) = 0.$$

By subtraction we obtain

$$(\lambda_n - \lambda_m)\int_0^L \phi^{(n)}(x)\phi^{(m)}(x)dx = 0.$$

This completes the proof. The above calculation holds as long as $\phi^{(n)'}(x)\phi^{(m)}(x)\Big|_0^L$ vanishes.

It is possible to derive from Theorem 2 but the following computation is necessary.

$$\int_0^L \sin\frac{(n-\frac{1}{2})\pi x}{L}\sin\frac{(m-\frac{1}{2})\pi x}{L}dx$$

$$= \int_0^L \left[\sin\frac{n\pi x}{L}\cos\frac{\pi x}{2L} - \cos\frac{n\pi x}{L}\sin\frac{\pi x}{2L}\right]\left[\sin\frac{m\pi x}{L}\cos\frac{\pi x}{2L} - \cos\frac{m\pi x}{L}\sin\frac{\pi x}{2L}\right]dx$$

$$= \frac{1}{2}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\left(1+\cos\frac{\pi x}{L}\right)dx + \frac{1}{2}\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\left(1-\cos\frac{\pi x}{L}\right)dx$$

$$-\frac{1}{2}\int_0^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\sin\frac{\pi x}{L}dx - \frac{1}{2}\int_0^L \cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\sin\frac{\pi x}{L}dx$$

$$= \frac{1}{2}\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx + \frac{1}{2}\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx$$

$$-\frac{1}{2}\int_0^L \cos\frac{(n+m)\pi x}{L}\cos\frac{\pi x}{L}dx - \frac{1}{2}\int_0^L \sin\frac{(n+m)\pi x}{L}\sin\frac{\pi x}{L}dx$$

$$= \frac{L}{4}\delta_{nm} + \frac{L}{4}\delta_{nm} - \frac{L}{4}\delta_{n+m,1} - \frac{L}{4}\delta_{n+m,1} = \frac{L}{2}\delta_{nm}.$$

Problem 3. (10 points) Let us consider the temperature in the steady state which is given as a solution to the heat equation $u_t = Ku_{zz}$ in the slab $0 < z < L$. If the boundary conditions are given by $u(0,t) = T_1$, $u_z(L,t) = \Phi_2$, then the steady-state temperature is $T_1 + \Phi_2 z$. Find the steady-state temperature for the bondary conditions $u_z(0,t) = \Phi_1$, $u(L,t) = T_2$.

Solution Since the solution is independent of t, let us write $U(z) = u(z,t)$. The general solution to $u_{zz} = 0 \quad \Leftrightarrow \quad U'' = 0$ is written as

$$U(z) = A + Bz.$$

Hence

$$u_z(0,t) = \Phi_1 \quad \Rightarrow \quad U'(0) = \Phi_1 \quad \Rightarrow \quad B = \Phi_1,$$

and

$$u(L,t) = T_2 \quad \Rightarrow \quad U(L) = T_2 \quad \Rightarrow \quad A + \Phi_1 L = T_2 \quad \Rightarrow \quad A = T_2 - \Phi_1 L.$$

Finally the steady-state solution is obtained as

$$u(z,t) = U(z) = T_2 - \Phi_1(L - z).$$

Problem 4. (10 points) Solve the initial-value problem for the heat equation $u_t = Ku_{zz}$ with the boundary conditions $u(0,t) = T$, $u_z(L,t) = \Phi$ and the initial condition $u(z,0) = T$, where K, Φ, T are positive constants.

Solution **Step 1** We find the steady-state solution $U(z)$ satisfying $U''(z) = 0$, $U(0) = T$, $U'(L) = \Phi$. By Problem 3, we obtain

$$U(z) = T + \Phi z.$$

Step 2 We introduce $v(z,t) = U(z) - u(z,t)$, which obeys $v_t = Kv_{zz}$, $v(0,t) = v_z(L,t) = 0$, $v(z,0) = T - U(z)$.

Step 3 We write $v(z,t) = \phi(z)T(t)$, where $\phi'' + \lambda\phi = 0$, $\phi(0) = \phi'(L) = 0$, $T' + \lambda KT = 0$. Using Problem 1, we obtain

$$\phi(z) = \phi^{(m)}(z) = \sin\frac{(m-\frac{1}{2})\pi z}{L}, \quad \lambda = \lambda^{(m)} = \left[\frac{(m-\frac{1}{2})\pi}{L}\right]^2, \quad m = 1, 2, \ldots.$$

Thus the general solution is

$$v(z,t) = \sum_{m=1}^{\infty} A_m \phi^{(m)}(z) e^{-\lambda^{(m)}Kt},$$

where A_m are constants. The coefficients are determined by

$$-\Phi z = \sum_{m=1}^{\infty} A_m \phi^{(m)}(z).$$

We multiply $\phi^{(n)}(z)$ on both sides and integrate over z:

$$\int_0^L (-\Phi) z \phi^{(n)}(z) dz = \int_0^L \sum_{m=1}^{\infty} A_m \phi^{(m)}(z)\phi^{(n)}(z) dz.$$

Using the orthogonality relations $\int_0^L \phi^{(m)}(z)\phi^{(n)}(z)dz = 0$ $(m \neq n)$ from Theorem 3 on the last page, we have

$$-\Phi \int_0^L z\phi^{(n)}(z)dz = A_n \int_0^L \left[\phi^{(n)}(z)\right]^2 dz.$$

Using Problem 1, we obtain

$$A_n = \frac{2}{L}(-\Phi)\left[\frac{L}{(n-\frac{1}{2})\pi}\right]^2 (-1)^{n+1} = \frac{2\Phi L}{(n-\frac{1}{2})^2\pi^2}(-1)^n.$$

Finally we obtain

$$\begin{aligned} u(z,t) &= U(z) + v(z,t) \\ &= T + \Phi z + \frac{2\Phi L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-\frac{1}{2})^2}\sin\frac{(n-1/2)\pi z}{L} e^{-[(n-1/2)\pi/L]^2 Kt}. \end{aligned}$$

(continued)

Formulae

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1, \quad \cosh(-x) = \cosh x, \quad \sinh(-x) = -\sinh x$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x, \quad \sinh(2x) = 2\sinh x \cosh x, \quad \tanh(2x) = \frac{2\tanh x}{1 + \tanh^2 x}$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}, \quad \sinh^2 x = \frac{\cosh 2x - 1}{2}, \quad 1 - \tanh^2 x = \mathrm{sech}^2 x = \frac{1}{\cosh^2 x}$$

$$\frac{d\cosh x}{dx} = \sinh x, \quad \frac{d\sinh x}{dx} = \cosh x, \quad \frac{d\tanh x}{dx} = \mathrm{sech}^2 x = \frac{1}{\cosh^2 x}$$

$$\begin{aligned}
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

$$\begin{aligned}
\cos A \cos B &= \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right] \\
\sin A \sin B &= \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right] \\
\sin A \cos B &= \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right] \\
\cos A \sin B &= \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]
\end{aligned}$$

$$\begin{aligned}
\cosh(A \pm B) &= \cosh A \cosh B \pm \sinh A \sinh B \\
\sinh(A \pm B) &= \sinh A \cosh B \pm \cosh A \sinh B \\
\tanh(A \pm B) &= \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}
\end{aligned}$$

$$\begin{aligned}
\cosh A \cosh B &= \frac{1}{2}\left[\cosh(A + B) + \cosh(A - B)\right] \\
\sinh A \sinh B &= \frac{1}{2}\left[\cosh(A + B) - \cosh(A - B)\right] \\
\sinh A \cosh B &= \frac{1}{2}\left[\sinh(A + B) + \sinh(A - B)\right] \\
\cosh A \sinh B &= \frac{1}{2}\left[\sinh(A + B) - \sinh(A - B)\right]
\end{aligned}$$

Theorems

Theorem 1. For $m, n = 1, 2, \cdots$, we have

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = L\delta_{nm},$$
$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = L\delta_{nm},$$
$$\int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0.$$

Theorem 2. For $m, n = 1, 2, \cdots$, we have

$$\int_{0}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \frac{L}{2}\delta_{nm},$$
$$\int_{0}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \frac{L}{2}\delta_{nm},$$
$$\int_{0}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \begin{cases} \dfrac{2Ln}{\pi\left(n^2 - m^2\right)} & \text{for odd } n + m, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3. Consider the Sturm-Liouville problem

$$[s(x)\phi'(x)]' + [\lambda\rho(x) - q(x)]\phi(x) = 0, \quad a < x < b,$$

where $\rho(x) > 0$, with the boundary conditions

$$\phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \quad \phi(b)\cos\beta + L\phi'(b)\sin\beta = 0,$$

where $L = b - a$, and $\alpha, \beta \in [0, \pi)$ are some parameters. Suppose that $\phi_1(x), \phi_2(x)$ are nontrivial solutions with different eigenvalues $\lambda_1 \neq \lambda_2$. Then the eigenfunctions are orthogonal with respect to the weight function $\rho(x)$, $a < x < b$:

$$\int_{a}^{b} \phi_1(x)\phi_2(x)\rho(x)dx = 0.$$

Theorem 4. For $m, n = 1, 2, \cdots$, we have

$$\int_{0}^{L_2}\int_{0}^{L_1} \sin \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} \sin \frac{m'\pi x}{L_1} \sin \frac{n'\pi y}{L_2} dxdy = \frac{L_1 L_2}{4}\delta_{mm'}\delta_{nn'}.$$