MATH 454 SECTION 002
MIDTERM 1

February 5, 2014, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- Formulae listed at the end can be used without proof.
- One side of a US letter size paper ($8.5'' \times 11''$) with notes is OK.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Problem	Points	Score
1	4	
2	6	
3	10	
4	10	
5	10	
TOTAL	40	

Problem 1. (4 points) Classify the following second-order equations into elliptic, parabolic, or hyperbolic (c, v, D are constants).

(a) $\dfrac{1}{v}\dfrac{\partial u}{\partial t}(x,t) = D\dfrac{\partial^2 u}{\partial x^2}(x,t)$, (b) $\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t)$, (c) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) u(x,y) = 0.$

Solution (a) parabolic, (b) hyperbolic, (c) elliptic.

Problem 2. (6 points) Consider the series below. Answer (a), (b), or (c).

$$\sum_{n=10}^{\infty} \frac{1}{n^2} = 0.?05166\ldots$$

The number in the first decimal place is (a) 0, (b) 1, (c) 2.

Solution We note that

$$\int_{10}^{\infty} \frac{1}{x^2} dx \le \sum_{n=10}^{\infty} \frac{1}{n^2} \le \int_{10}^{\infty} \frac{1}{(x-1)^2} dx.$$

The integrals are calculated as

$$\int_{10}^{\infty} \frac{1}{x^2} dx = \frac{1}{10} = 0.1, \quad \int_{10}^{\infty} \frac{1}{(x-1)^2} dx = \int_{9}^{\infty} \frac{1}{y^2} dy = \frac{1}{9} = 0.111\ldots.$$

Therefore,

$$0.1 \le \sum_{n=10}^{\infty} \frac{1}{n^2} \le 0.111\ldots.$$

Hence‡,

$$? = 1.$$

‡ Using Problem 5, we obtain

$$\sum_{n=10}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - \frac{1}{2^2} - \frac{1}{3^2} - \cdots - \frac{1}{9^2} = \frac{\pi^2}{6} - 1 - \frac{1}{4} - \frac{1}{9} - \cdots - \frac{1}{81} = 0.105166\ldots.$$

Problem 3. (10 points) Find the Fourier sine series for $f(x) = \cos x$, $0 < x < \pi$.

Solution We extend $f(x)$ to the interval $-\pi < x < \pi$ by defining

$$f_O(x) = \begin{cases} \cos x & \text{for } 0 < x < \pi, \\ 0 & \text{for } x = 0, \\ -\cos x & \text{for } -\pi < x < 0. \end{cases}$$

We consider the Fourier series of this odd function f_O:

$$f_O(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos(nx) + B_n \sin(nx)\right), \quad -\pi < x < \pi.$$

Since $f_O(x)$ is odd, we have A_0 and $A_n = 0$. We obtain

$$\begin{aligned} B_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_O(x) \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} \cos(x) \sin(nx) dx \\ &= \frac{1}{\pi} \int_0^{\pi} \left[\sin(nx + x) + \sin(nx - x)\right] dx. \end{aligned}$$

If $n = 1$, we have

$$B_1 = \frac{1}{\pi} \int_0^{\pi} \sin(2x) dx = 0.$$

For $n \geq 2$ we obtain

$$B_n = \frac{1}{\pi} \left[\frac{-\cos((n+1)x)}{n+1} + \frac{-\cos((n-1)x)}{n-1}\right]\Bigg|_0^{\pi} = \frac{2}{\pi} \frac{n[1 + (-1)^n]}{n^2 - 1}.$$

Therefore we obtain

$$f_O(x) = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{n[1 + (-1)^n]}{n^2 - 1} \sin(nx), \quad -\pi < x < \pi.$$

The Fourier sine series is obtained as

$$\begin{aligned} \cos x &= \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{n[1 + (-1)^n]}{n^2 - 1} \sin(nx), \quad 0 < x < \pi, \\ &= \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{m}{4m^2 - 1} \sin(2mx), \quad 0 < x < \pi. \end{aligned}$$

(continued)

Problem 4. (10 points) Find the separated solutions $u(x,t)$ of the heat equation $u_t - u_{xx} = 0$ in the region $0 < x < L$, $t > 0$, that satisfy the boundary conditions $u_x(0,t) = u_x(L,t) = 0$.

Solution We look for a separated solution $u(x,t) = X(x)T(t)$. We get

$$\frac{T'}{T} - \frac{X''}{X} = 0.$$

By introducing the separation constant λ, we obtain§

$$T'(t) = \lambda T(t), \quad X''(x) = \lambda X(x).$$

Thus, for three cases $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$, we have‖

$$u(x,t) = \begin{cases} \left(A_1 \cosh(kx) + A_2 \sinh(kx)\right) e^{k^2 t} & \text{for } \lambda = k^2,\ k > 0, \\ A_1 x + A_2 & \text{for } \lambda = 0, \\ \left(A_1 \cos(lx) + A_2 \sin(lx)\right) e^{-l^2 t} & \text{for } \lambda = -l^2,\ l > 0. \end{cases}$$

For $\lambda > 0$, the boundary conditions imply $A_1 = A_2 = 0$. Similarly for $\lambda = 0$, we can conclude $A_1 = 0$. Only the case of $\lambda < 0$ has nontrivial solutions. From the boundary conditions, we have $A_2 = 0$ and $\sin(lL) = 0$. Hence $lL = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Let C be a constant. We obtain

$$u(x,t) = C \cos \frac{n\pi x}{L} e^{-(n\pi/L)^2 t} \quad (n = 0, 1, 2, \ldots).$$

§ The introduction of λ is not unique. So, $T'(t) = -\lambda T(t)$, $X''(x) = -\lambda X(x)$, and $T'(t) = 2\lambda T(t)$, $X''(x) = 2\lambda X(x)$, etc. are all fine. The final solution $u(x,t)$ will be the same.

‖ In the case of $\lambda > 0$, we can also write $\left(A_1 e^{kx} + A_2 e^{-kx}\right) e^{k^2 t}$. In the case of $\lambda < 0$, we can also write $\left(A_1 e^{ilx} + A_2 e^{-ilx}\right) e^{-l^2 t}$.

(continued)

Problem 5. (10 points) Find $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots$. (*Hint:* $f(x)=x$, $-\pi<x<\pi$.)

Solution Since x is an odd function, the Fourier series of x is written as

$$x=\sum_{n=1}^{\infty} B_n \sin(nx).$$

Here,

$$\begin{aligned}
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)dx \\
&= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx)\bigg|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos(nx)dx\right] \\
&= \frac{2}{\pi}\left(-\frac{\pi}{n}\right)\cos(n\pi) \\
&= \frac{2}{n}(-1)^{n+1}.
\end{aligned}$$

According to Parseval's formula, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx=\frac{1}{2}\sum_{n=1}^{\infty} B_n^2.$$

The left-hand side is calculated as

$$\text{LHS}=\frac{1}{2\pi}\left.\frac{x^3}{3}\right|_{-\pi}^{\pi}=\frac{\pi^2}{3}.$$

The right-hand side is calculated as

$$\text{RHS}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{2}{n}(-1)^{n+1}\right)^2=2\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Therefore,

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

(continued)

Formulae

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\cosh^2 x - \sinh^2 x = 1, \quad \cosh(-x) = \cosh x, \quad \sinh(-x) = -\sinh x$$
$$\cosh(2x) = \cosh^2 x + \sinh^2 x, \quad \sinh(2x) = 2\sinh x \cosh x, \quad \tanh(2x) = \frac{2\tanh x}{1 + \tanh^2 x}$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}, \quad \sinh^2 x = \frac{\cosh 2x - 1}{2}, \quad 1 - \tanh^2 x = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x}$$
$$\frac{d\cosh x}{dx} = \sinh x, \quad \frac{d\sinh x}{dx} = \cosh x, \quad \frac{d\tanh x}{dx} = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x}$$

$$\begin{aligned}
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

$$\begin{aligned}
\cos A \cos B &= \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right] \\
\sin A \sin B &= \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right] \\
\sin A \cos B &= \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right] \\
\cos A \sin B &= \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]
\end{aligned}$$

$$\begin{aligned}
\cosh(A \pm B) &= \cosh A \cosh B \pm \sinh A \sinh B \\
\sinh(A \pm B) &= \sinh A \cosh B \pm \cosh A \sinh B \\
\tanh(A \pm B) &= \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}
\end{aligned}$$

$$\begin{aligned}
\cosh A \cosh B &= \frac{1}{2}\left[\cosh(A + B) + \cosh(A - B)\right] \\
\sinh A \sinh B &= \frac{1}{2}\left[\cosh(A + B) - \cosh(A - B)\right] \\
\sinh A \cosh B &= \frac{1}{2}\left[\sinh(A + B) + \sinh(A - B)\right] \\
\cosh A \sinh B &= \frac{1}{2}\left[\sinh(A + B) - \sinh(A - B)\right]
\end{aligned}$$