MATH 454 SECTION 002
FINAL

April 30, 2014, Instructor: Manabu Machida

Name: ____

- To receive full credit you must show all your work.
- Formulae listed at the end can be used without proof.
- Theorems listed at the end can be used without proof.
- You can also use results from other problems, e.g., you can use Problem 1 when you solve Problem 2.
- Both sides of a US letter size paper ($8.5'' \times 11''$) with notes is OK.
- You can use the back side of a paper if you need. Indicate where your calculation jumps.
- **NO CALCULATOR, SMARTPHONE, BOOKS, or OTHER NOTES.**

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
TOTAL	80	

Problem 1. (10 points) Write the general solution $u(\rho, \varphi)$ of Laplace's equation $\nabla^2 u = 0$ in the cylindrical region $1 < \rho < 2$. (*Hint:* if necessary, you can use the fact that $\Phi''(\varphi) + \mu\Phi(\varphi) = 0$, $\Phi(-\pi) = \Phi(\pi)$, $\Phi'(-\pi) = \Phi'(\pi)$ is solved as $\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi$, $\mu = m^2$, $m = 0, 1, 2, \ldots$.)

Solution We solve

$$\nabla^2 u = u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\varphi\varphi} = 0.$$

Assuming a solution of the form $u(\rho, \varphi) = R(\rho)\Phi(\varphi)$ and using the separation constant $\lambda = -\Phi''/\Phi$, we have

$$\Phi'' + \lambda\Phi = 0, \quad \Phi(-\pi) = \Phi(\pi), \quad \Phi'(-\pi) = \Phi'(\pi), \qquad R'' + \frac{1}{\rho}R' - \frac{\lambda}{\rho^2}R = 0.$$

We obtain $\Phi(\varphi) = A_m \cos m\varphi + B_m \sin m\varphi$, $\lambda = m^2$, $m = 0, 1, 2, \ldots$. When $m = 0$, two linearly independent solutions to $R'' + (1/\rho)R' = 0$ are $1, \ln\rho$. For $m \neq 0$, two solutions are found as $R(\rho) = \rho^m, \rho^{-m}$. Therefore the general solution is obtained as

$$u(\rho, \varphi) = A_0 + B_0 \ln\rho + \sum_{m=1}^{\infty} \rho^m (A_m \cos m\varphi + B_m \sin m\varphi) + \sum_{m=1}^{\infty} \rho^{-m}(C_m \cos m\varphi + D_m \sin m\varphi),$$

where $A_0, B_0, A_m, B_m, C_m, D_m$ are constants.

Problem 2. (10 points) Consider a function $u(x, y, z)$. We want to use cylindrical coordinates ρ, φ, z instead of x, y, z. By using $\rho^2 = x^2 + y^2$ and $y = \rho \sin\varphi$, we obtain

$$u_x = \boxed{\mathrm{A}}\, u_\rho - \boxed{\mathrm{B}}\, u_\varphi.$$

Find $\boxed{\mathrm{A}}$ and $\boxed{\mathrm{B}}$. In this way we obtain $\Delta u = u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\varphi\varphi} + u_{zz}$.

Solution We note that

$$\rho^2 = x^2 + y^2 \quad \Rightarrow \quad 2\rho \frac{\partial \rho}{\partial x} = 2x, \quad 2\rho \frac{\partial \rho}{\partial y} = 2y$$
$$\Rightarrow \quad \frac{\partial \rho}{\partial x} = \frac{x}{\rho} = \cos\varphi, \quad \frac{\partial \rho}{\partial y} = \frac{y}{\rho} = \sin\varphi,$$

and

$$y = \rho \sin\varphi \quad \Rightarrow \quad 0 = \frac{\partial \rho}{\partial x} \sin\varphi + \rho \cos\varphi \frac{\partial \varphi}{\partial x} = \cos\varphi \sin\varphi + \rho \cos\varphi \frac{\partial \varphi}{\partial x}$$
$$\Rightarrow \quad \frac{\partial \varphi}{\partial x} = -\frac{\sin\varphi}{\rho}.$$

We have

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial \varphi} \frac{\partial \varphi}{\partial x} = \cos\varphi \frac{\partial u}{\partial \rho} - \frac{\sin\varphi}{\rho} \frac{\partial u}{\partial \varphi}.$$

Therefore

$$\boxed{\mathrm{A}} = \cos\varphi, \qquad \boxed{\mathrm{B}} = \frac{\sin\varphi}{\rho}.$$

Problem 3. (10 points) Find $u(x,t)$ for the heat equation

$$\begin{cases} u_t = K u_{xx}, & t > 0, \quad -\infty < x < \infty, \\ u = e^{-x^2}, & t = 0, \quad -\infty < x < \infty. \end{cases}$$

Solution The solution $u(x,t)$ is written as

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi K t}} e^{-(x-x')^2/4Kt} e^{-x'^2} dx'.$$

We note that

$$-\frac{(x-x')^2}{4Kt} - x'^2 = -\frac{4Kt+1}{4Kt}\left(x' - \frac{x}{4Kt+1}\right)^2 - \frac{x^2}{4Kt+1}.$$

Therefore we have

$$u(x,t) = \frac{1}{\sqrt{4\pi K t}} e^{-x^2/(4Kt+1)} \int_{-\infty}^{\infty} \exp\left[-\frac{4Kt+1}{4Kt}\left(x' - \frac{x}{4Kt+1}\right)^2\right] dx'.$$

Using the Gaussian integral we obtain

$$u(x,t) = \frac{1}{\sqrt{4Kt+1}} e^{-x^2/(4Kt+1)}.$$

(continued)

Problem 4. (10 points) Solve

$$\begin{cases} tu_t + xu_x + u = 0, & t > 1, \quad -\infty < x < \infty, \\ u = x, & t = 1, \quad -\infty < x < \infty. \end{cases}$$

Solution Let us introduce s and τ. We have

$$\begin{cases} \dfrac{dt}{ds} = t, & s > 0, \\ t = 1, & s = 0, \end{cases}$$

and

$$\begin{cases} \dfrac{dx}{ds} = x, & s > 0, \\ x = \tau, & s = 0, \end{cases}$$

By solving the equations we obtain

$$t = e^s, \qquad x = \tau e^s.$$

We note that

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} = tu_t + xu_x = -u.$$

Therefore we have

$$\begin{cases} \dfrac{du}{ds} + u = 0, & s > 0, \\ u = \tau, & s = 0. \end{cases}$$

We obtain

$$u = e^{-s}\tau.$$

Since

$$s = \ln t, \qquad \tau = \frac{x}{t},$$

finally we obtain

$$u(x,t) = \frac{1}{t}\frac{x}{t} = \frac{x}{t^2}.$$

Problem 5. (10 points) Let us consider the vibrating (circular) membrane problem (i.e., the edges are fixed) in the case where the radius is a and $u(\rho,\varphi,0)=0$. The general solution to $u_{tt}=c^2\nabla^2 u$ is written as

$$u(\rho,\varphi,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A_{mn}\cos m\varphi+B_{mn}\sin m\varphi)\sin\frac{ctx_n^{(m)}}{a},$$

where $J_m(x_n^{(m)})=0$, $x_n^{(m)}>0$, and A_{mn},B_{mn} are constants. Find the solution $u(\rho,\varphi,t)$ when $u_t(\rho,\varphi,0)=J_5(\rho x_1^{(5)}/a)\sin(5\varphi)$, $0<\rho<a$.

Solution We introduce $x=\rho/a$. To satisfy the condition $u_t(\rho,\varphi,0)=J_5(xx_1^{(5)})\sin(5\varphi)$, A_{mn},B_{mn} must satisfy

$$J_5(xx_1^{(5)})\sin(5\varphi)=\sum_{m'=0}^{\infty}\sum_{n'=1}^{\infty}\frac{cx_{n'}^{(m')}}{a}J_{m'}(xx_{n'}^{(m')})\left(A_{m'n'}\cos m'\varphi+B_{m'n'}\sin m'\varphi\right).$$

If we multiply $\sin m\varphi$, $m=1,2,\ldots$, and integrate on both sides over φ, we obtain

$$J_5(xx_1^{(5)})\pi\delta_{m5}=\pi\sum_{n'=1}^{\infty}\frac{cx_{n'}^{(m)}}{a}J_m(xx_{n'}^{(m)})B_{mn'},\tag{1}$$

where we used the orthogonality relations. Similarly by multiplying $\cos m\varphi$, $m=0,1,2,\ldots$, we obtain

$$0=\pi\sum_{n'=1}^{\infty}\frac{cx_{n'}^{(m)}}{a}J_m(xx_{n'}^{(m)})A_{mn'}.\tag{2}$$

We then multiply (1) and (2) by $J_m(xx_n^{(m)})x$ and integrate over x. Using the orthogonality relations we have

$$\frac{1}{2}J_6(x_1^{(5)})^2\delta_{m5}\delta_{n1}=\frac{cx_n^{(m)}}{a}\frac{1}{2}J_{m+1}(x_n^{(m)})^2B_{mn},\qquad 0=\frac{cx_n^{(m)}}{a}\frac{1}{2}J_{m+1}(x_n^{(m)})^2A_{mn}.$$

Hence $A_{mn}=B_{mn}=0$ except $B_{51}=a/(cx_1^{(5)})$. Finally we obtain

$$u(\rho,\varphi,t)=\frac{a}{cx_1^{(5)}}J_5\left(\frac{\rho x_1^{(5)}}{a}\right)\sin 5\varphi\sin\frac{ctx_1^{(5)}}{a}.$$

(continued)

Problem 6. (10 points) Let $\{x_n\}$ be the nonnegative solutions to $J_m(x_n) = 0$, where $m \geq 0$. We have

$$\int_0^1 J_m(xx_{n_1})J_m(xx_{n_2})xdx = \frac{1}{2}J_{m+1}(x_{n_1})^2\delta_{n_1n_2}.$$

Let us show that the right-hand side is 0 when $n_1 \neq n_2$. The proof is shown below. What are $\boxed{\text{A}}$, $\boxed{\text{B}}$, and $\boxed{\text{C}}$?

Step 1 We note that $\frac{d^2J_m(x)}{dx^2} + \frac{1}{x}\frac{dJ_m(x)}{dx} + \left(1 - \frac{m^2}{x^2}\right)J_m(x) = 0$. Hence,

$$\frac{d}{dx}\left(x\frac{dy_1(x)}{dx}\right) + \left(xx_{n_1}^2 - \frac{m^2}{x}\right)y_1(x) = 0, \tag{3}$$

$$\frac{d}{dx}\left(x\frac{dy_2(x)}{dx}\right) + \left(xx_{n_2}^2 - \frac{m^2}{x}\right)y_2(x) = 0, \tag{4}$$

where $y_i(x) = J_m(xx_{n_i})$ $(i = 1, 2)$.

Step 2 We multiply (3) by $\boxed{\text{A}}$ and multiply (4) by $\boxed{\text{B}}$, and integrate both sides over x.

Step 3 By subtracting the resulting equations we obtain

$$(y_1'y_2 - y_1y_2')\Big|_{x=1} + \boxed{\text{C}}\int_0^1 xy_1(x)y_2(x)dx = 0.$$

The first term vanishes due to the boundary conditions. Since $n_1 \neq n_2$, thus, the orthogonality relations are proved.

Solution

$$\boxed{\text{A}} = y_2(x), \qquad \boxed{\text{B}} = y_1(x), \qquad \boxed{\text{C}} = x_{n_1}^2 - x_{n_2}^2.$$

Problem 7. (10 points) Let $f(s) = 0$ for $-1 < s < \frac{1}{2}$ and $f(s) = 1$ for $\frac{1}{2} < s < 1$. Find the expansion of $f(s)$ in a series of Legendre polynomials. (*Hint:* $\int_{-1}^{1} P_k(s)P_j(s)ds = \frac{2}{2k+1}\delta_{kj}$.)

Solution We write

$$f(s) = \sum_{k=0}^{\infty} A_k P_k(s), \qquad -1 < s < 1,$$

where A_k are constants to be determined. We multiply $P_j(s)$ and integrate over s:

$$\int_{-1}^{1} f(s)P_j(s)ds = \sum_{k=0}^{\infty} A_k \int_{-1}^{1} P_k(s)P_j(s)ds.$$

The left-hand side is obtained as follows. For $j = 0$, we have

$$\text{LHS} = \int_{1/2}^{1} P_0(s)ds = \int_{1/2}^{1} ds = \frac{1}{2}.$$

For $j \geq 1$, we have

$$\text{LHS} = \int_{1/2}^{1} P_j(s)ds = \int_{1/2}^{1} \frac{-1}{j(j+1)} \frac{d}{ds}\left[(1-s^2)\frac{d}{ds}P_j(s)\right] ds$$
$$= \frac{-1}{j(j+1)} (1-s^2)\frac{d}{ds}P_j(s)\bigg|_{1/2}^{1} = \frac{1}{j(j+1)}\frac{3}{4}\frac{dP_j}{ds}(\frac{1}{2}).$$

The right-hand side is obtained as

$$\text{RHS} = \sum_{k=0}^{\infty} A_k \int_{-1}^{1} P_k(s)P_j(s)ds = \sum_{k=0}^{\infty} A_k \frac{2}{2k+1}\delta_{kj} = \frac{2A_j}{2j+1}.$$

Therefore we obtain

$$A_0 = \frac{1}{4}, \qquad A_j = \frac{3(2j+1)}{8j(j+1)} P_j'\left(\frac{1}{2}\right) \quad (j \geq 1).$$

That is,

$$f(s) = \frac{1}{4} + \frac{3}{8}\sum_{k=1}^{\infty} \frac{2k+1}{k(k+1)} P_k'\left(\frac{1}{2}\right) P_k(s), \qquad -1 < s < 1.$$

Problem 8. (10 points) Find $u(x,t)$.

$$\begin{cases} u_t = Ku_{xx}, & t>0,\ 0<x<L, \\ u=0, & t>0,\ x=0,L, \\ u=\delta(x-1), & t=0,\ 0<x<L. \end{cases}$$

Solution By assuming the form $u(x,t)=\phi(x)T(t)$ and introducing the separation constant $\lambda=-\phi''/\phi$, we have

$$\phi''+\lambda\phi=0, \quad \phi(0)=\phi(L)=0, \qquad T'+\lambda KT=0.$$

We obtain

$$\phi(x)=\phi_n(x)=\sin\frac{n\pi x}{L}, \quad \lambda=\lambda_n=\left(\frac{n\pi}{L}\right)^2, \quad n=1,2,\ldots, \qquad T(t)=e^{-\lambda Kt}.$$

Note that

$$\int_0^L \phi_n(x)\phi_m(x)dx=\frac{L}{2}\delta_{nm}.$$

Thus the general solution is given by

$$u(x,t)=\sum_{n=1}^{\infty}A_n\phi_n(x)e^{-\lambda_n Kt}.$$

The initial condition is written as

$$\delta(x-1)=\sum_{n=1}^{\infty}A_n\phi_n(x).$$

If $L<1$, then we have $A_n=0$ and $u(x,t)=0$. Hereafter we assume $L>1$. We multiply $\phi_m(x)$ on both sides and integrate over x.

$$\int_0^L \delta(x-1)\phi_m(x)dx=\int_0^L\sum_{n=1}^{\infty}A_n\phi_n(x)\phi_m(x)dx.$$

We obtain $\phi_m(1)=A_m\frac{L}{2}$, and

$$A_m=\frac{2}{L}\phi_m(1).$$

Therefore we obtain

$$u(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi}{L}\sin\frac{n\pi x}{L}e^{-(n\pi/L)^2Kt}.$$

(continued)

Alternative Solution Let us extend $\delta(x-1)$ as an odd $2L$-periodic function by setting

$$f_O(x) = \begin{cases} \delta(x - 2mL - 1), & 2mL < x < (2m+1)L, \\ 0, & x = 2mL, \quad (2m+1)L, \quad (2m+2)L, \\ -\delta(-x + (2m+2)L - 1), & (2m+1)L < x < (2m+2)L, \end{cases}$$

where $m = 0, \pm 1, \pm 2, \ldots$. Note that $f_O(x + 2L) = f_O(x)$ for all x. Then we have

$$u(x,t) = \int_{-\infty}^{\infty} G(x,x';t) f_O(x') dx' = \sum_{m=-\infty}^{\infty} \left\{ \int_{2mL}^{(2m+1)L} + \int_{(2m+1)L}^{(2m+2)L} \right\} G(x,x';t) f_O(x') dx'.$$

We obtain

$$u(x,t) = \int_0^L G_L(x,x';t)\delta(x'-1)dx',$$

where

$$G_L(x,x';t) = \sum_{m=-\infty}^{\infty} \left[G(x, x' + 2mL; t) - G(x, -x' + (2m+2)L; t) \right],$$

and

$$G(x,x';t) = \frac{1}{\sqrt{4\pi K t}} e^{-(x-x')^2/4Kt}.$$

If $L < 1$, then we have $u(x,t) = 0$. If $L > 1$, we obtain

$$\begin{aligned} u(x,t) &= G_L(x,1;t) \\ &= \frac{1}{\sqrt{4\pi K t}} \sum_{m=-\infty}^{\infty} \left[e^{-(x-2mL-1)^2/4Kt} - e^{-(x-(2m+2)L+1)^2/4Kt} \right]. \end{aligned}$$

Formulae

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\cosh^2 x - \sinh^2 x = 1, \quad \cosh(-x) = \cosh x, \quad \sinh(-x) = -\sinh x$$
$$\cosh(2x) = \cosh^2 x + \sinh^2 x, \quad \sinh(2x) = 2\sinh x \cosh x, \quad \tanh(2x) = \frac{2\tanh x}{1 + \tanh^2 x}$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}, \quad \sinh^2 x = \frac{\cosh 2x - 1}{2}, \quad 1 - \tanh^2 x = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x}$$
$$\frac{d\cosh x}{dx} = \sinh x, \quad \frac{d\sinh x}{dx} = \cosh x, \quad \frac{d\tanh x}{dx} = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x}$$

$$\begin{aligned}
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

$$\begin{aligned}
\cos A \cos B &= \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right] \\
\sin A \sin B &= \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right] \\
\sin A \cos B &= \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right] \\
\cos A \sin B &= \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]
\end{aligned}$$

$$\begin{aligned}
\cosh(A \pm B) &= \cosh A \cosh B \pm \sinh A \sinh B \\
\sinh(A \pm B) &= \sinh A \cosh B \pm \cosh A \sinh B \\
\tanh(A \pm B) &= \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}
\end{aligned}$$

$$\begin{aligned}
\cosh A \cosh B &= \frac{1}{2}\left[\cosh(A + B) + \cosh(A - B)\right] \\
\sinh A \sinh B &= \frac{1}{2}\left[\cosh(A + B) - \cosh(A - B)\right] \\
\sinh A \cosh B &= \frac{1}{2}\left[\sinh(A + B) + \sinh(A - B)\right] \\
\cosh A \sinh B &= \frac{1}{2}\left[\sinh(A + B) - \sinh(A - B)\right]
\end{aligned}$$

Theorems

Theorem 1. For $m, n = 1, 2, \cdots$, we have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = L\delta_{nm},$$
$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = L\delta_{nm},$$
$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0.$$

Theorem 2. For $m, n = 1, 2, \cdots$, we have

$$\int_{0}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{L}{2}\delta_{nm},$$
$$\int_{0}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \frac{L}{2}\delta_{nm},$$
$$\int_{0}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} \dfrac{2Ln}{\pi\left(n^2 - m^2\right)} & \text{for odd } n+m, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3. Consider the Sturm-Liouville problem

$$[s(x)\phi'(x)]' + [\lambda\rho(x) - q(x)]\phi(x) = 0, \quad a < x < b,$$

where $\rho(x) > 0$, with the boundary conditions

$$\phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \quad \phi(b)\cos\beta + L\phi'(b)\sin\beta = 0,$$

where $L = b - a$, and $\alpha, \beta \in [0, \pi)$ are some parameters. Suppose that $\phi_1(x), \phi_2(x)$ are nontrivial solutions with different eigenvalues $\lambda_1 \neq \lambda_2$. Then the eigenfunctions are orthogonal with respect to the weight function $\rho(x)$, $a < x < b$:

$$\int_{a}^{b} \phi_1(x)\phi_2(x)\rho(x)dx = 0.$$

Theorem 4. For $m, n = 1, 2, \cdots$, we have

$$\int_{0}^{L_2}\int_{0}^{L_1} \sin\frac{m\pi x}{L_1}\sin\frac{n\pi y}{L_2}\sin\frac{m'\pi x}{L_1}\sin\frac{n'\pi y}{L_2}dxdy = \frac{L_1L_2}{4}\delta_{mm'}\delta_{nn'}.$$

Cylindrical and spherical coordinates

$$x = \rho\cos\varphi, \qquad y = \rho\sin\varphi, \qquad z = z,$$

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta.$$

Bessel's equation

$$J''_m(x) + \frac{1}{x}J'_m(x) + \left(1 - \frac{m^2}{x^2}\right)J_m(x) = 0.$$

The Legendre equation

$$\left[(1-s^2)P'_k(s)\right]' + k(k+1)P_k(s) = 0.$$

Fourier transform

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(\mu)e^{i\mu x}d\mu, \qquad \tilde{f}(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\mu x}dx.$$

Gaussian integral

$$\int_{-\infty}^{\infty} e^{-a(x-b)^2}dx = \sqrt{\frac{\pi}{a}}, \qquad a > 0.$$

Green's functions

The solution to

$$\begin{cases} u_t - Ku_{xx} = h(x,t), & t > 0, \quad -\infty < x < \infty, \\ u = f(x), & t = 0, \quad -\infty < x < \infty, \end{cases}$$

is given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x,x';t)f(x')dx' + \int_0^t \int_{-\infty}^{\infty} G(x,x';t-s)h(x',s)dx'ds,$$

where

$$G(x,x';t) = \frac{1}{\sqrt{4\pi Kt}}e^{-(x-x')^2/(4Kt)}.$$